HOMOLOGY OF TYPES IN MODEL THEORY II: A HUREWICZ THEOREM FOR STABLE THEORIES

JOHN GOODRICK, BYUNGHAN KIM, AND ALEXEI KOLESNIKOV

Abstract. This is a sequel to the paper [6] ‘Homology of types in model theory I: Basic concepts and connections with type amalgamation’. We compute the group $H_2$ for strong types in stable theories and show that any profinite abelian group can occur as the group $H_2$ in the model-theoretic context.

The work described in this paper was originally inspired by Hrushovski’s discovery [8] of striking connections between amalgamation properties and definable groupoids in models of a stable first-order theory.

Amalgamation properties have already been much studied by researchers in simple theories. The Independence Theorem, or 3-amalgamation, was used to construct canonical bases for types in such theories [11][7], and in [2], Hrushovski’s group configuration theorem for stable theories was generalized to simple theories under the assumption of 4-amalgamation over sets containing models. In [10], the $n$-amalgamation hierarchy was studied systematically. See Section 1 below for a precise definition of $n$-amalgamation.

In [8], Hrushovski showed that if a stable theory fails 3-uniqueness, then there must exist a groupoid whose sets of objects and morphisms, as well as the composition of morphisms, are definable in the theory. In [4], an explicit construction of such a groupoid was given and it was shown in [5] that the group of automorphisms of each object of such a groupoid must be abelian profinite. The morphisms in the groupoid construction in [4] arise as equivalence classes of “paths”, defined in a model-theoretic way. In some sense, the groupoid construction paralleled that of the construction of a fundamental groupoid in a topological space. Thus it seemed natural to ask whether it is possible to define the notion of a homology group in model-theoretic context and, if yes, would the homology group be linked to the group described in [4, 5].

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In the companion paper [6], we defined homology groups $H_n(p)$ linked to a complete type $p$ in any simple (or even rosy) theory and showed that if $T$ has $k$-amalgamation for every $k$ between 1 and $n+2$, then $H_n(p) = 0$ for any complete type $p$. This left open the question of the converse, whether the triviality of all homology groups could imply $n$-amalgamation for every $n$. In this paper, we make a step in this direction, showing that a failure of 4-amalgamation in a stable theory implies the nontriviality of $H_2(p)$.

We have a conjecture as to what $H_n(p)$ should be for $n > 2$ – see Question 2.3 below. However, we ran into an obstacle when trying to extend our technique to higher homology groups: while failures of 4-amalgamation in a stable theory are always witnessed by definable non-eliminable groupoids (as explained in detail below in Section 2), we do not know a corresponding fact for higher amalgamation. Conjecturally, we believe that failures of $(n+2)$-amalgamation in a stable theory with $(n+1)$-complete amalgamation should be witnessed by some sort of definable $n$-groupoids (that is, a kind of $n$-category), but we do not know how to prove this. Also note that we make use the stability of the theory $T$ to show that the group $\Gamma_2(p)$ in our main theorem is well-defined, as well as in the construction of the groupoid in Proposition 2.7.

In Section 1, we briefly recall the definition of the homology group $H_n(p)$ from [6] and the definitions of the type amalgamation properties. In Section 2, we state and prove the main result on $H_2(p)$ in stable theories (Theorem 2.1). In the final section, Section 3, we build a family of examples showing that the abstract groups which can occur as $H_2(p)$ in a stable theory are precisely the profinite abelian groups (in a sense this complements the main result of Section 2, which implies that $H_2(p)$ is necessarily profinite).

Throughout this paper, we assume that $T$ is a complete stable theory and that $T = T^{eq}$. As usual, $\mathcal{C}$ denotes a large, sufficiently saturated model of $T$ (the “monster model”) and all elements and sets are assumed to come from $\mathcal{C}$. For general background on simple and stable theories, the reader is referred to the book [15], which explains nonforking, imaginaries, and much more.

1. Definition of the homology groups

In this section we recall the definition of the homology groups from our other paper [6]. In the terminology of that paper, we will only use here the “set simplices” and not the “type simplices” (which yield
equivalent definitions of $H_n(p)$. We will not need to use any of the results from [6], only the definitions.

If $s$ is a set, then we consider the power set $\mathcal{P}(s)$ of $s$ to be a category with a single inclusion map $\iota_{u,v} : u \to v$ between any pair of subsets $u$ and $v$ with $u \subseteq v$. A subset $X \subseteq \mathcal{P}(s)$ is called downward-closed if whenever $u \subseteq v \in X$, then $u \in X$. In this case we consider $X$ to be a full subcategory of $\mathcal{P}(s)$. An example of a downward-closed collection that we will use below is $\mathcal{P}^-(s) := \mathcal{P}(s) \setminus \{s\}$.

**Definition 1.1.** Let $A$ be a small subset of $\mathcal{C}$. By $\mathcal{C}_A$ we denote the category of all subsets (not necessarily algebraically closed) of $\mathcal{C}$ of size no more that $\kappa_0$, where morphisms are partial elementary maps over $A$ (that is, fixing $A$ pointwise).

**Definition 1.2.** If $A = \text{acl}(A)$ is a small subset of $\mathcal{C}$ and $p \in S(A)$, then a closed independent set-functor in $p$ is a functor $f : X \to \mathcal{C}_A$ such that:

1. For some finite $s \subseteq \omega$, $X$ is a downward-closed subset of $\mathcal{P}(s)$;
2. For any $t \in X$, $|f(t)| \leq (|A| + \aleph_0)^{|T|}$;
3. If $u \subseteq t \in X$, then the image $f^u_i := f(\iota_{u,i})$ fixes $A$ pointwise;
4. $f(\emptyset) = A$;
5. For every $i \in s$, $f(\{i\})$ (if it is defined) is $\text{acl}(A \cup a_i)$ for some realization $a_i$ of $p$; and
6. For all non-empty $u \in X$, we have that $f(u) = \text{acl}(A \cup \bigcup_{i \in u} f^{|i|}_i(\{i\}))$ and the set $\{f^{|i|}_i(\{i\}) : i \in u\}$ is $A$-independent.

**Definition 1.3.** Let $n \geq 0$ be a natural number and $p \in S(A)$. An $n$-simplex in $p$ is a closed independent set-functor functor $f : \mathcal{P}(s) \to \mathcal{C}$ in $p$ for some set $s \subseteq \omega$ with $|s| = n + 1$. The set $s$ is called the support of $f$, or supp($f$).

Let $S_n(p)$ denote the collection of all $n$-simplices in $p$. Then put $S(p) := \bigcup_n S_n(p)$.

Let $C_n(p)$ denote the free abelian group generated by $S_n(p)$; its elements are called $n$-chains in $p$, or $n$-chains in $p$. Similarly, we define $C(p) := \bigcup_n C_n(p)$. The support of a chain $c$ is the union of the supports of all the simplices that appear in $c$ with a nonzero coefficient.

**Definition 1.4.** If $n \geq 1$ and $0 \leq i \leq n$, then the $i$th boundary operator $\partial_i^n : C_n(p) \to C_{n-1}(p)$ is defined so that if $f$ is an $n$-simplex in $p$ with domain $\mathcal{P}(s)$, where $s = \{s_0, \ldots, s_n\}$ with $s_0 < \ldots < s_n$, then

$$\partial_i^n(f) = f \upharpoonright \mathcal{P}(s \setminus \{s_i\})$$

and extended linearly to a group map on all of $C_n(p)$. 
If $n \geq 1$ and $0 \leq i \leq n$, then the boundary map $\partial_n : C_n(p) \to C_{n-1}(p)$ is defined by the rule

$$\partial_n(c) = \sum_{0 \leq i \leq n} (-1)^i \partial_i^n(c).$$

We write $\partial_i$ and $\partial$ for $\partial_i^n$ and $\partial_n$, respectively, if $n$ is clear from context.

**Definition 1.5.** The kernel of $\partial_n$ is denoted $Z_n(p)$, and its elements are called $(n)$-cycles. The image of $\partial_{n+1}$ in $C_n(p)$ is denoted $B_n(p)$. The elements of $B_n(p)$ are called $(n)$-boundaries.

It can be shown (by the usual combinatorial argument) that $B_n(p) \subseteq Z_n(p)$, or more briefly, “$\partial_n \circ \partial_{n+1} = 0$.” Therefore we can define simplicial homology groups in the type $p$:

**Definition 1.6.** The $n$th (simplicial) homology group of $p \in S(A)$ is $H_n(p) := Z_n(p)/B_n(p)$.

The definition above only makes sense for $n > 0$. Since this paper is only concerned with $H_2(p)$, we refer the curious reader to [6] for a discussion of what $H_0(p)$ might mean (we propose two different definitions there, but in either case it turns out that $H_0(p)$ gives no information about the type $p$).

Finally, we define the amalgamation properties. We use the convention that $[n]$ denotes the $(n + 1)$-element set $\{0, 1, \ldots, n\}$.

**Definition 1.7.** Let $n \geq 1$.

1. $p \in S(A)$ has $n$-amalgamation if for any closed independent set-functor $f : \mathcal{P}([-n]) \to \mathcal{C}_A$ in the type $p$, there is an $(n-1)$-simplex $g$ in $p$ such that $g \supseteq f$.
2. $p \in S(A)$ has $n$-uniqueness if for any closed independent set-functor $f : \mathcal{P}([-n]) \to \mathcal{C}_A$ in $p$ and any two $(n-1)$-simplices $g_1$ and $g_2$ in $p$ extending $f$, there is a natural isomorphism $F : g_1 \to g_2$ such that $F \upharpoonright \text{dom}(f)$ is the identity.
3. The theory $T$ has $n$-amalgamation (or $n$-existence) just in case all of its types $p \in S(A)$ have this property.
4. A type $p \in S(A)$ or a theory $T$ has $n$-complete amalgamation or $n$-CA if it has $k$-amalgamation for every $k$ with $1 \leq k \leq n$.

It was observed in [8] that for a stable theory, 3-uniqueness is equivalent to 4-existence. A link between $n$-amalgamation and the homology groups was shown in [6], where the following was proved (as Corollary 3.7 (2)):

**Fact 1.8.** If $n \geq 3$ and $p$ has $n$-CA, then $H_{n-2}(p) = 0$. 
2. Computing $H_2(p)$ (the “Hurewicz theorem”)

We assume throughout this section that $p$ is a strong type (without loss of generality, over the empty set). We will prove that the homology group $H_2(p)$ is isomorphic to a certain automorphism group $\Gamma_2(p)$ defined below. This can be thought of as an analogue of Hurewicz’s theorem in algebraic topology, which says that for a path connected topological space $X$, the first homology group $H_1(X)$ is isomorphic to the abelianization of the homotopy group $\pi_1(X)$. Just as there is a higher-dimensional version of Hurewicz’s theorem for $H_n(X)$ under the hypothesis that $X$ is $(n - 1)$-connected, we hope that there is a higher-dimensional generalization of our result under the hypothesis that the theory $T$ has $(n + 1)$-complete amalgamation. In other words, maybe $n$-CA is analogous to a topological connectedness property.

Throughout this section, “$a$” denotes the algebraic closure of an element $a$ (possibly together with a fixed ambient parameter set), considered as a possibly infinite ordered tuple, but the choice of ordering will not play any important role in what follows. Implicit in the argument below is that if $a \equiv a_0$, then there are orderings $\overline{a}, \overline{a_0}$ of their algebraic closures such that $\overline{a} \equiv \overline{a_0}$. Moreover, $\text{Aut}(A/B)$ denotes the group of elementary maps from $A$ onto $A$ fixing $B$ pointwise.

First, suppose that $C = \{a_i : i \in s\}$ is an independent set of realizations of the type $p$. Pick some $a$ realizing $p$ such that $a \downarrow C$, and let

$$\tilde{a}_s := \overline{a}_s \cap \text{dcl} \left( \bigcup_{i \in s} \overline{a_i a_{s \setminus \{i\}}} \right).$$

Note that since $T$ is stable, by stationarity, the set $\tilde{a}_s$ does not depend on the particular choice of $a$.

Fix some integer $n \geq 2$, and let $\{a_0, \ldots, a_{n-1}\}$ be an independent set of $n + 1$ realizations of $p$. Recall our notation that $[k] = \{0, \ldots, k\}$, so that $\tilde{a}_{[n-1]} = \tilde{a}_{\{0, \ldots, n-1\}}$. Let

$$B_n = \bigcup_{0 \leq i \leq n-1} \tilde{a}_{\{0, \ldots, i, \ldots, n-1\}}.$$

Finally, we let $\Gamma_n(p) = \text{Aut}(\tilde{a}_{[n-1]}/B_n)$.

Note that $\tilde{a}_{[n-1]}$ is a subset of $\text{acl}(a_0, \ldots, a_{n-1})$, so $\Gamma_n(p)$ is a quotient of the full automorphism group $\text{Aut}(\tilde{a}_{[n-1]}/B_n)$ (namely, the quotient of the subgroup of all automorphisms fixing $\tilde{a}_{[n-1]}$ pointwise).

Now we can state the main result of this section:

**Theorem 2.1.** If $T$ is stable, $p$ is stationary, and $(a, b) \models p^{(2)}$, then $H_2(p) \cong \Gamma_2(p)$. 

An immediate consequence of this theorem plus Fact 1.8 above is:

**Corollary 2.2.** If $T$ is a stable theory, then $T$ has 3-uniqueness (or equivalently, $T$ has 4-amalgamation) if and only if for every strong type $p$ (possibly over a set $A$ of extra parameters), $H_2(p) = 0$.

**Question 2.3.** If $T$ is stable with $(n+1)$-complete amalgamation, then is $H_n(p)$ isomorphic to $\Gamma_n(p)$?

### 2.1. Preliminaries on definable groupoids.

Here we review some material from [4] and [5] on definable groupoids that we need for the proof of Theorem 2.1. We also make a minor correction to a lemma from [4] and set some notation that will be used later. Recall that we assume $T$ is stable.

We know from [4] that in a stable theory, failures of 3-uniqueness (or equivalently, of 4-amalgamation) are linked with type-definable connected groupoids which are not retractable. (See that paper for definitions of these terms.) It turns out that the groupoid $G$ associated to such a failure of 3-uniqueness can even be assumed to have abelian “vertex groups” $\text{Mor}_G(a, a)$ (this is proved in Section 2 of [5]).

Given an acl(\emptyset)-definable connected groupoid $G'$ such that the groups $G'_a := \text{Mor}_{G'}(a, a)$ are all finite and abelian, we can define canonical isomorphisms between any two groups $G'_a$ and $G'_b$ via conjugation by some (any) $h \in \text{Mor}_{G'}(a, b)$. Therefore we can quotient $\bigcup_{a \in \text{Ob}(G')} G'_a$ by this system of commuting automorphisms to get a binding group $G'_b$, and note that $G'$ can be thought of as a subset of acl(eq)(\emptyset). In fact, even if the mentioned groupoid $G$ is only type-definable (more precisely, relatively definable), we can still associate the binding group $G$ with a subset of acl(\emptyset): first find a definable connected extension $G'$ of $G$ in which $G$ is a full faithful subcategory, then apply this argument to $G'$. If $h \in G_a$, let $[h]_{G'}$ be the corresponding element of $G$ (so identify $G$ and $G'$).

Next we recall from [5] the definition of a “full symmetric witness to the failure of 3-uniqueness.” For the present paper, we modify the definition slightly so that a full symmetric witness is a tuple $W$ containing a formula $\theta$ witnessing the key property. (Later we will need to keep track of this formula).

**Definition 2.4.** A full symmetric witness to non-3-uniqueness (over an algebraically closed set $A$) is a tuple $(a_0, a_1, a_2, f_{01}, f_{12}, f_{02}, \theta(x, y, z))$ such that $a_0, a_1, a_2$ and $f_{01}, f_{12}, f_{02}$ are finite tuples, $\{a_0, a_1, a_2\}$ is independent over $A$, $\theta(x, y, z)$ is a formula over $A$, and:

1. $f_{ij} \in \overline{a_{ij}}$;
2. $f_{01} \notin \text{dcl}(\overline{a_0a_1})$;
The following (proved in [5]) is the key technical point saying that we have “enough” symmetric witnesses:

**Proposition 2.5.** If \( T \) does not have 3-uniqueness, then there is a set \( A \) and a full symmetric witness to non-3-uniqueness over \( A \).

In fact, if \((a_0, a_1, a_2)\) is the beginning of a Morley sequence and \( f \) is any element of \( a_{01} \cap \text{dcl}(a_{02}, a_{12}) \) which is not in \( \text{dcl}(a_0, a_1) \), then there is some full symmetric witness \((a'_0, a'_1, a'_2, f', g, h, \theta)\) such that \( f \in \text{dcl}(f') \) and \( a_i \in \text{dcl}(a'_i) \subseteq \overline{a_i} \) for \( i = 0, 1, 2 \).

The next lemma states a crucial point in the construction of type-definable groupoids from witnesses to the failure of 3-uniqueness. This was not isolated as a lemma in [4], though the idea was there.

**Lemma 2.6.** If \((a_0, a_1, a_2, f_{01}, f_{12}, f_{02}, \theta(x, y, z))\) is a full symmetric witness, and if \( f \equiv_{a_0} f_{01} \) and \( g \equiv_{a_1} f_{12} \), then
\[
(f, g, \overline{a_0}, \overline{a_1}, \overline{a_2}) \equiv (f_{01}, f_{12}, \overline{a_0}, \overline{a_1}, \overline{a_2}).
\]

**Proof.** By clause (5) in the definition of a full symmetric witness, \((f, \overline{a_0}, \overline{a_1}) \equiv (f_{01}, \overline{a_0}, \overline{a_1})\) and \((g, \overline{a_1}, \overline{a_2}) \equiv (f_{12}, \overline{a_1}, \overline{a_2})\). It follows (by the stationarity of types over \( \overline{a_1} \)) that
\[
(f, g, \overline{a_0}, \overline{a_1}, \overline{a_2}) \equiv (f_{01}, g, \overline{a_0}, \overline{a_1}, \overline{a_2})
\]
and
\[
(f_{01}, g, \overline{a_0}, \overline{a_1}, \overline{a_2}) \equiv (f_{01}, f_{12}, \overline{a_0}, \overline{a_1}, \overline{a_2}),
\]
and the lemma follows. \( \square \)

Given any full symmetric witness to the failure of 3-uniqueness, we can construct from it a connected, type-definable groupoid:

**Proposition 2.7.** Let \( W = (a_0, a_1, a_2, f, g, h, \theta(x, y, z)) \) be a full symmetric witness (over \( \emptyset \)). Then from \( W \) we can construct a connected groupoid \( G_W \) which is type definable over \( \text{acl}(\emptyset) \) and has the following properties:

1. The objects of \( G_W \) are the realizations of the type \( p = \text{stp}(a_1) \).
2. Let \( SW_{a_0, a_1} := \{ f' : f' \equiv_{a_0, a_1} f \} \).
   There is a bijection \( f \mapsto [f]_{G_W}^{a_0, a_1} \) from \( SW_{a_0, a_1} \) onto \( \text{Mor}_{G_W} (a_0, a_1) \) which is definable over \( (a_0, a_1) \).
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(3) If \( f_0, f_1 \in \text{Mor}_G(a_0, a_1) \), then \( f_0 \equiv_{a_0, a_1} f_1 \).
(4) The “vertex groups” \( \text{Mor}_{G_W}(a, a) \) are finite and abelian.

Proof. We build \( G_W \) using a slight modification of the construction described in subsection 2.2 of [4]. The problem with the construction in that paper is that Remark 2.8 there is incorrect as stated: in general, just because \((a, b, f) \equiv (a_0, a_1, f_01) \equiv (b, c, g)\), it does not follow that \((a, b, c, f, g) \equiv (a_0, a_2, f_01, f_12)\) (if the tuples \(a_i\) are not algebraically closed, \(f_01\) may contain elements of \(\text{acl}(a_0) \setminus \text{dcl}(a_0)\), and this could cause \(\text{tp}(a, f, g)\) to differ from \(\text{tp}(a_0, f_01, f_12)\)). However, Lemma 2.6 and the fact that we are using a full symmetric witness eliminates this problem.

In particular, if \((a, b, f) \equiv (a_0, a_1, f_01) \equiv (b, c, g)\), then there is a unique element “\(g \circ f\)” such that \(|= \theta(f, g, g \circ f)\) and \((a, c, g \circ f) \equiv (a_0, a_2, f_02)\).

From here, everything else in the construction of the type-definable groupoid \( G = G_W \) in [4] works. Property (1) of the proposition follows directly from the construction, and property (2) is just like Lemma 2.14 of [4]. Because of the definable bijection in (2), any two morphisms in \(\text{Mor}_G(a_0, a_1)\) have the same type, yielding (3). Finally, property (4) is Corollary 2.7 of [5]. \(\square\)

Next, here is a more detailed version Proposition 2.15 from [5], which we will use later.

**Proposition 2.8.** Suppose that \((a_0, a_1, a_2, f_01, f_12, f_02, \theta)\) is a full symmetric witness, and \(G\) is the associated type-definable groupoid as in Proposition 2.7. If \(SW\) is the set \(\{f' : \text{tp}(f'/a_0, a_1) = \text{tp}(f_01/a_0, a_1)\}\), then there is a group isomorphism

\[
\psi^0_G : \text{Mor}_G(a_1, a_1) \to \text{Aut}(SW/a_0, a_1)
\]

defined by the rule: if \(g \in \text{Mor}_G(a_1, a_1)\), then \(\psi^0_G(g)\) is the unique element \(\sigma \in \text{Aut}(SW/a_0, a_1)\) which induces the same left action on \(\text{Mor}_G(a_0, a_1)\) as left composition by \(g\).

Furthermore, the inclusion map \(\text{Aut}(SW/a_0, a_1) \to \text{Aut}(SW/a_0, a_1)\) is surjective, so we actually have an isomorphism

\[
\psi_G : \text{Mor}_G(a_1, a_1) \to \text{Aut}(SW/a_0, a_1)
\]

Proof. The “Furthermore ...” clause was not in Proposition 2.15 of [5], but it follows from the fact that the witness is fully symmetric: if \(f'\) is any element of \(SW\), then clause (5) of the definition of a symmetric witness implies that \(\text{tp}(f'/\overline{a_0}, \overline{a_1}) = \text{tp}(f_01/\overline{a_0}, \overline{a_1})\), and so there is an element \(\sigma \in \text{Aut}(SW/\overline{a_0}, \overline{a_1})\) such that \(\sigma(f_01) = f'\). This means that there are at least \(|\text{Mor}_G(a_1, a_0)|\) different elements in \(\text{Aut}(SW/\overline{a_0}, \overline{a_1});\)
but, by the first part of the proposition, there are only $|\text{Mor}_G(a_1, a_0)|$ elements in $\text{Aut}(SW/a_0, a_1)$. Since this is a finite set, the injective inclusion map $\text{Aut}(SW/\overline{a_0}, \overline{a_1}) \to \text{Aut}(SW/a_0, a_1)$ is surjective.

\[\therefore\]

2.2. Proof of Theorem 2.1. We assume throughout the proof that $p \in S(\emptyset)$ and $\text{acl}(\emptyset) = \text{dcl}(\emptyset)$ (since we can add constants for the parameters of $p$ if necessary). It follows directly from the definitions that if $p = \text{tp}(a)$ and $p' = \text{tp}(a')$ where $a$ and $a'$ are interalgebraic, then $H_n(p) = H_n(p')$. Therefore, by Proposition 2.5 above, we may assume that there are some $(a_0, a_1, a_2)$ realizing $p^{(3)}$ and a full symmetric witness $(a_0, a_1, a_2, f_{01}, f_{12}, f_{02}, \theta(x, y, z))$ to this failure. We pick one such witness which we fix throughout the proof. Note that we assume the $f_{ij}$’s to be finite tuples, and also that there may be more than one such witness (which is the interesting case). We assume that there is at least one such witness, since otherwise $H_2(p)$ and $\Gamma_2(p)$ are both trivial.

As already observed in [5], the symmetric witnesses in the type $p$ form a directed system. To make this more precise, pick some $(a_0, a_1, a_2)$ realizing $p^{(3)}$ (which we fix for the remainder of the subsection). Now we build a directed system of full symmetric witnesses as follows:

Claim 2.9. There is a directed partially ordered set $\langle I, \leq \rangle$ and an $I$-indexed collection of symmetric witnesses $\langle W_i : i \in I \rangle$ such that for any $i$ and $j$ in $I$:

1. $W_i = (a_i^0, a_i^1, a_i^2, f_i^{01}, f_i^{12}, f_i^{02}, \theta_i^1(x_i, y_i, z_i))$ is a full symmetric witness to failure of 3-uniqueness;
2. $a_i^0, a_i^1 \in \text{dcl}(f_i^{01})$;
3. if $i \leq j$, then $f_i^{01} \in \text{dcl}(f_j^{01})$, $a_i^0 \in \text{dcl}(a_j^0) \subseteq \overline{a_0}$, and $a_i^0 a_0^i \equiv a_i^0 a_0^i$;

and satisfying the maximality conditions

\[
\overline{a_{i_0, i_1}} = \text{dcl} \left( \bigcup_{i \in I} f_{01}^0 \right)
\]

and

\[
\overline{a_0} = \text{dcl} \left( \bigcup_{i \in I} a_i^0 \right).
\]

Proof. We will build the partial ordering $\langle I, \leq \rangle$ as the union of a countable chain of partial orderings $I_0 \subseteq I_1 \subseteq \ldots$ such that for any $i, j \in I_n$ there is a $k \in I_{n+1}$ such that $i \leq k$ and $j \leq k$. Then the partial ordering $I = \bigcup_{n \in \omega} I_n$ will be directed.

First, let
\[ W_i^0 = \langle a_0^i, a_1^i, a_2^i, f_{01}^i, f_{12}^i, f_{02}^i, \theta^i(x_i, y_i, z_i) : i \in I_0 \rangle \]

be any collection of full symmetric witnesses large enough to satisfy the two maximality conditions in the statement of the Claim, where \( I_0 \) is a trivial partial ordering in which no two distinct elements are comparable. For the induction step, suppose that we have the partial ordering \( I_n \) (for some \( n \in \omega \)) and full symmetric witnesses \((a_0^i, \ldots, \theta^i(x_i, y_i, z_i))\) for each \( i \in I_n \). First, we can build a partial ordering \( I_{n+1} \) by adding one new point immediately above every pair of points in \( I_n \) and such that any two new points in \( I_{n+1} \setminus I_n \) are incomparable. Then by Proposition 2.5, there are corresponding full symmetric witnesses \((a_0^i, \ldots, \theta^i)\) for each \( i \in I_{n+1} \setminus I_n \) such that if \( j \) and \( k \) are less than or equal to \( i \), then \( f_{01}^j, f_{01}^k \in dcl(f_{01}^i) \) and \( a_0^j, a_0^k \in dcl(a_0^i) \). Similarly, we can ensure condition (2) (that \( a_0^i, a_1^i \in dcl(f_{01}^i) \)) for the new symmetric witnesses.

Let \( p_i = \text{stp}(a_0^i) \) and \( G^*_i \) be the type-definable groupoid constructed from the full symmetric witness \( W_i \) as in Proposition 2.7 above. So \( \text{Ob}(G_i) = p_i(\mathcal{C}) \) and the groups \( \text{Mor}_{G_i}(a_0^i, a_0^i) \) are finite and abelian, and we have the corresponding finite abelian groups \( G^*_i \). As explained above, can (and will) assume that the groups \( G^*_i \) are subsets of \( acl(\emptyset) \).

For any \( i \in I \), let \( SW_i \) be the set of all realizations of \( \text{tp}(f_{01}^i/a_0^i, a_1^i) \) (which is a finite set). If \( (a, b) \models p_i^{(2)} \), let \( SW(a, b) \) be the image of \( SW_i \) under an automorphism of \( \mathcal{C} \) that maps \( (a_0^i, a_1^i) \) to \( (a, b) \). Recall from Proposition 2.7 that we have a definable map \( f \mapsto [f]_{G_i}^{a, b} \) from \( SW(a, b) \) onto \( \text{Mor}_{G_i}(a, b) \), from which we can define an inverse map \( g \mapsto \langle g \rangle_{G_i}^{a, b} \) from \( \text{Mor}_{G_i}(a, b) \) to \( SW(a, b) \). For convenience, we will write these maps as \( \langle \cdot \rangle_i^{a, b, m} \) and \( \langle \cdot \rangle_i^{a, b, n} \) or even just \( \langle \cdot \rangle_i^{a, b, m} \) and \( \langle \cdot \rangle_i^{a, b, n} \) when \( (a, b) \) is clear from context.

**Lemma 2.10.** There are systems of relatively \( \emptyset \)-definable functions
\( \langle \pi_{j,i} : i \leq j, j \in I \rangle \) and \( \langle \tau_{j,i} : i \leq j, j \in I \rangle \) (that is, they are the intersection of an \( \emptyset \)-definable relation with the product of their domain and range) such that whenever \( i \leq j \),

1. \( \tau_{j,i} : p_j(\mathcal{C}) \to p_i(\mathcal{C}) \)
2. \( \pi_{j,i} : \bigcup_{(a,b)\models p_i^{(2)}} SW(a, b) \to SW(\tau_{j,i}(a), \tau_{j,i}(b)) \)
3. \( \tau_{j,i}(a_0^i) = a_0^i \)
4. \( \tau_{j,i}(a_1^i) = a_1^i \)
5. \( \pi_{j,i} \) maps \( SW_j \) onto \( SW_i \), and
6. whenever \( i \leq j \leq k \),
(7) \( \tau_{j,i} \circ \tau_{k,j} = \tau_{k,i} \) and

(8) \( \pi_{j,i} \circ \pi_{k,j} = \pi_{k,i} \).

**Proof.** First, the maps \( \tau_{j,i} \) can be constructed satisfying (1), (3), and (4) using the facts that \( a_0^j \in \text{dcl}(a_0^j) \), \( a_0^j \in \text{dcl}(a_1^j) \), and \( a_0^j a_0^j \equiv a_1^j a_1^j \) (from clause (3) of Claim 2.9). Now if \( i \leq j \leq k \), since \( \tau_{k,i}(x) = \tau_{j,i} \circ \tau_{k,j}(x) \) is true for \( x = a_k \), this holds for every \( x \) in the domain of \( \tau_{k,i} \) (because the domain is a complete type), and so (7) holds.

If \( i \leq j \), then since \( f_{01}^j \in \text{dcl}(f_{01}^j) \), we can pick a relatively definable map \( \pi_{j,i} \) such that \( \pi_{j,i}(f_{01}^j) = f_{01}^j \). As before, if \( i \leq j \leq k \), since \( \pi_{k,i}(x) = \pi_{j,i} \circ \pi_{k,j}(x) \) holds for \( x = f_{01}^k \), it holds for any \( x \) in any of the sets \( \text{SW}(a,b) \) for \( (a,b) \models p^{(2)} \), so (8) holds.

 Ideally, we would like the functions \( \pi_{j,i} \) and \( \tau_{j,i} \) of Lemma 2.10 to induce a commuting system of functors \( F_{j,i} : G_j^* \to G_i^* \), which we could use to construct and inverse limit \( \mathcal{G} \) of \( \langle G_i^* : i \in I \rangle \). This is essentially what we do, and we will then show that the group \( \text{Mor}_G(\mathcal{G}_0, \mathcal{G}_0) \) is isomorphic to both \( H_2(p) \) and \( \Gamma_2(p) \). However, first we need to modify the formulas \( \theta_i^* \) slightly for this to be true.

The key to making all of this work is the following technical lemma.

**Lemma 2.11.** There is a family of formulas \( \langle \theta_i(x_i, y_i, z_i) : i \in I \rangle \) such that

1. \( W_i \) is still a full symmetric witness with \( \theta_i^*(x_i, y_i, z_i) \) replaced by \( \theta_i(x_i, y_i, z_i) \) and \( f_{02}^i \) replaced by another element of \( \text{SW}(a_0^j, a_1^j) \), and

2. whenever \( i \leq j \), \( f \in \text{SW}(a_0^j, a_1^j) \), \( g \in \text{SW}(a_1^j, a_2^j) \), and \( h \in \text{SW}(a_0^j, a_2^j) \), then

\[ \models \theta_j(f, g, h) \rightarrow \theta_i(\pi_{j,i}(f), \pi_{j,i}(g), \pi_{j,i}(h)). \]

**Proof.** Recall from above that \( (a_0, a_1, a_2) \) realizes \( p^{(3)} \). We use Zorn’s Lemma to find a maximal subset \( J \subseteq I \) and formulas \( \theta_j(x_j, y_j, z_j) \) for each \( j \in J \) satisfying the following properties:

3. For every \( j \in J \), there are elements \( f_j, g_j, \) and \( h_j \) such that \( (a_0^j, a_1^j, a_2^j, f_j, g_j, h_j, \theta_j(x_j, y_j, z_j)) \) is a full symmetric witness; and

4. If \( j_1, \ldots, j_n \in J \) and \( (a_0^{j_1}, a_1^{j_1}, a_2^{j_1}, f_{j_1}, g_{j_1}, h_{j_1}, \theta_{j_1})(s = 1, \ldots, n \) and if \( f_{j_1} \ldots f_{j_n} \equiv g_{j_1} \ldots g_{j_n} \), then \( f_{j_1} \ldots f_{j_n} \equiv h_{j_1} \ldots h_{j_n} \).

**Claim 2.12.** \( J = I \).
Proof. Suppose towards a contradiction that there is some $k \in J \setminus J$. Let $F_J = \langle f^\alpha \rangle$ be a (possibly infinite) tuple listing every element of $\bigcup_{j \in J} SW(a^j_0, a^j_1)$, and let $a^i_j$ (for $i \in \{0, 1, 2\}$) be a tuple listing $\{a^i_j : j \in J\}$, ordered the same way as $F_J$. Pick $f_k \in SW(a^0_k, a^1_k)$, and then pick $G_J = \langle g^\alpha \rangle$ and $g_k$ such that $F_J f_k a^i_j a^j_1 \equiv G_J g_k a^i_j a^j_2$. Note that $g^\alpha \in SW(a^1_j, a^2_j)$. Next pick a tuple $H_J = \langle h^\alpha \rangle$ such that if $f^\alpha \in SW(a^0_0, a^1_1)$, then $\models \theta_J(f^\alpha, g^\alpha, h^\alpha)$. The element $h_j$ is well-defined because if it happens that $f^\alpha$ is also in $SW(a^j_0, a^j_1)$ for some $j' \neq j$, and if we let $h'$ be the unique element such that $\models \theta_J(f^\alpha, g^\alpha, h')$, then by property (4), the fact that $f^\alpha f^\alpha \equiv g^\alpha g^\alpha$ implies that $f^\alpha f^\alpha \equiv h^\alpha h'$, and so $h' = h^\alpha$.

By the assumption (4) on the set $J$, $F_J \equiv H_J$. Finally, pick an element $h_k$ such that $F_J f_k \equiv H_J h_k$. By Corollary 2.14 of [5], there is a formula $\theta_k$ such that $(a^0_k, a^1_k, a^2_k, f_k, g_k, h_k, \theta_k)$ is a full symmetric witness.

We claim that $J \cup \{k\}$ with $\theta_k$ satisfies condition (4) above, contradicting the maximality condition on the set $J$. Indeed, suppose that $j_1, \ldots, j_n \in J$, and the tuples

$$(a^i_j, a^j_k, f_{j_s}, g_{j_s}, h_{j_s}, \theta_{j_s})$$

(for $s = 1, \ldots, n$) and

$$(a^i_k, a^j_k, f_{k_s}, g_{k_s}, h_{k_s}, \theta_{k_s})$$

are full symmetric witnesses, and that $f_{j_1} \ldots f_{j_n}, f'_{k_s} \equiv g_{j_1} \ldots g_{j_n}, g'_{k_s}$. By the stationarity of $tp(f'_{k_s}/a^2)$ and $tp(g'_{k_s}/a^2)$, there is a $\sigma \in Aut(\mathcal{C}/a^0, a^1, a^2)$ such that $\sigma(f'_{k_s}) = f_k$ and $\sigma(g'_{k_s}) = g_k$ for the $f_k$ and $g_k$ from the previous paragraph. By the same argument and using the fact that $F_J \equiv G_J$, we can also assume that if $\sigma((f_{j_1}, \ldots, f_{j_n})) = (f'^{\alpha_1}, \ldots, f'^{\alpha_n})$, then $\sigma((g_{j_1}, \ldots, g_{j_n})) = (g'^{\alpha_1}, \ldots, g'^{\alpha_n})$ (that is, the two tuples $(f_{j_1}, \ldots, f_{j_n})$ and $(g_{j_1}, \ldots, g_{j_n})$ map to corresponding subtypes of $F_J$ and $G_J$). It follows that $\sigma(h'_{k_s}) = h_k$ and $\sigma(h_{j_s}) = h'^{\alpha_s}$ for each $s$ between 1 and $n$. By our construction, $f^{\alpha_1} \ldots f^{\alpha_n} f_k \equiv h^{\alpha_1} \ldots h'^{\alpha_n} h_k$, and so by taking preimages under $\sigma$, we get that $f_{j_1} \ldots f_{j_n}, f'_{k_s} \equiv h_{j_1} \ldots k_{j_n}, h'_{k_s}$.

Finally, we check that condition (2) of the lemma holds for our new formulas $\theta_i$. Suppose that $i \leq j$, $f \in SW(a^0_i, a^1_i)$, $g \in SW(a^1_j, a^2_j)$, $h \in SW(a^2_0, a^2_j)$, and $\models \theta_j(f, g, h)$. Let $f_0 = \pi_{j,i}(f)$, and pick $g_0$ such that $f f_0 \equiv g g_0$. Then $g_0 = \pi_{j,i}(g)$. Finally, let $h_0$ be the unique element such that $\models \theta_j(f_0, g_0, h_0)$. By condition (4) above, $h h_0 \equiv f f_0$, and so $h_0 = \pi_{j,i}(h)$. Thus $\models \theta_i(\pi_{j,i}(f), \pi_{j,i}(g), \pi_{j,i}(h))$ as desired.
For each $i \in I$, let $G_i$ be the type-definable groupoid obtained from the symmetric witness $W_i$ with the modified formula $\theta_i$ from Lemma 2.11. Once again, the groups $\text{Mor}_{G_i}(a,a)$ are finite and abelian for any $a \in \text{Ob}(G_i)$, so we have the corresponding finite abelian groups $G_i$ which we consider as subsets of $\text{acl}(\emptyset)$.

Lemma 2.13. If $i \leq j \in I$, $(a,b,c) \models p_j^{(3)}$, $f \in \text{Mor}_{G_j}(a,b)$, and $g \in \text{Mor}_{G_j}(b,c)$, then

$$\pi_{j,i}(\langle g \circ f \rangle) = \pi_{j,i}(\langle g \rangle) \circ \pi_{j,i}(\langle f \rangle)$$

(where $\circ$ denotes composition in the groupoids $G_j$ and $G_i$).

Proof. By Proposition 2.12 of [5], $\theta_j$ defines groupoid composition between generic triples of objects in $G_j$, so

$$\models \theta_j((f)_j, (g)_j, (g \circ f)_j).$$

So by Lemma 2.11,

$$\models \theta_i(\pi_{j,i}((f)_j), \pi_{j,i}((g)_j), \pi_{j,i}((g \circ f)_j)).$$

By Proposition 2.12 again, the Lemma follows. \hfill \dasharrow

If $i \leq j \in I$ and $(a,b) \models p_j^{(2)}$, then because $SW(\tau_{j,i}(a), \tau_{j,i}(b)) \subseteq \text{dcl}(SW(a,b))$, we have a canonical surjective group map

$$\rho_{j,i}^{a,b} : \text{Aut}(SW(a,b)/\pi, \bar{b}) \to \text{Aut}(SW(\tau_{j,i}(a), \tau_{j,i}(b))/\pi, \bar{b}),$$

and these maps satisfy the coherence condition that $\rho_{k,i}^{a,b} = \rho_{j,i}^{a,b} \circ \rho_{k,j}^{a,b}$ whenever $i \leq j \leq k$. We will write “$\rho_{j,i}$” for $\rho_{j,i}^{a,b}$ if $(a,b)$ is clear from context.

For every $i \in I$, we also have a group isomorphism $\psi_i : \text{Mor}_{G_i}(a_i, a_i) \to \text{Aut}(SW_i/\pi_0, \bar{a}_i)$ as in Proposition 2.8 above.

The following is similar to Claim 2.17 of [5], except that here we have expanded this to a system of groupoid maps.

Lemma 2.14. For every $i \leq j \in I$, we define a map $\chi_{j,i} : G_j \to G_i$ by the rules:

1. If $a \in \text{Ob}(G_i)$, then $\chi_{j,i}(a) = \tau_{j,i}(a)$; and
2. If $f \in \text{Mor}_{G_j}(a,b)$, $c \models p_j((a,b)$, and $f = g \circ h$ for some $g \in \text{Mor}_{G_j}(c,b)$ and $h \in \text{Mor}_{G_j}(a,c)$, then

$$\chi_{j,i}(f) = \pi_{j,i}((h)_j) \circ \pi_{j,i}((g)_j).$$

Then the maps $\chi_{j,i}$ satisfy:

3. $\chi_{j,i}$ is a well-defined functor;
4. $\chi_{j,i}$ is full: every morphism in $\text{Mor}(G_i)$ is in the image of $\chi_{j,i}$;
5. $\chi_{j,i}$ is type-definable over $\text{acl}(\emptyset)$;
Proof. Suppose that $f \in \text{Mor}_{G_i}(a, b)$. To check that $\chi_{j,i}(f)$ is well-defined (and does not depend on the choices of $c, g$, and $h$), first note that given $c \models p_{ij}(a, b)$ and morphisms $g, h$ as in (2), the morphism $h$ is uniquely determined from $f$ and $g$, and for any other $g' \in \text{Mor}_{G_j}(c, b)$, $\text{tp}(f, g/a, b, c) = \text{tp}(f, g'/a, b, c)$ (by Lemma 2.6). So the choices of $f$ and $g$ do not matter once we have picked $c$, and the choice of $c$ does not matter by the stationarity of $p_{ij}$.

To show that $\chi_{j,i}$ is a functor, suppose that $a, b, c$ realize $p_{ij}$, $f \in \text{Mor}_{G_i}(a, b)$, and $g \in \text{Mor}_{G_j}(b, c)$. To compute the images of $f$ and $g$, we pick $(d, e) \models p_{ij}(a, b, c)$ and $f_0 \in \text{Mor}_{G_j}(a, d), f_1 \in \text{Mor}_{G_j}(d, b), g_0 \in \text{Mor}_{G_j}(b, e)$, and $g_1 \in \text{Mor}_{G_j}(e, c)$ such that $f = f_1 \circ f_0$ and $g = g_1 \circ g_0$. Then by the definition given in (2) of the Lemma,

$$\chi_{j,i}(f) = [\pi_{j,i}(\langle g_1 \circ g_0 \circ f_1 \rangle_j)]_i \circ [\pi_{j,i}(\langle f_0 \rangle_j)]_i.$$ 

By Lemma 2.13 twice, this equals

$$[\pi_{j,i}(\langle g \rangle_j)]_i \circ [\pi_{j,i}(\langle g_0 \rangle_j)]_i \circ [\pi_{j,i}(\langle f_1 \rangle_j)]_i \circ [\pi_{j,i}(\langle f_0 \rangle_j)]_i.$$ 

But the composition of the first two terms above equals $\chi_{j,i}(g)$ and the composition of the third and fourth terms equals $\chi_{j,i}(f)$, so $\chi_{j,i}(g \circ f) = \chi_{j,i}(g) \circ \chi_{j,i}(f)$.

Suppose that $a, b \in \text{Ob}(G_i)$ and $f \in \text{Mor}_{G_i}(a, b)$. Pick some $c \models p_i|(a, b)$, and pick $g \in \text{Mor}_{G_i}(c, b)$ and $h \in \text{Mor}_{G_i}(a, c)$ such that $f = g \circ h$. Since

$$\langle (g) \rangle_i, c, b \rangle \equiv (f_0^i, a_0^i, a_1^i) \equiv \langle (h) \rangle_i, a, c \rangle,$$

we can find elements $g'$ and $h'$ such that $\pi_{j,i}(g') = \langle g \rangle_i$ and $\pi_{j,i}(h') = \langle h \rangle_i$. Let $f' = [g']_j \circ [h']_j$. Unwinding the definitions, we see that

$$\chi_{j,i}(f') = [\pi_{j,i}(g')]_i \circ [\pi_{j,i}(h')]_i = [\langle g \rangle_i]_i \circ [\langle h \rangle_i]_i = g \circ h = f.$$ 

This establishes that the functor $\chi_{j,i}$ is full.
The fact that \( \chi_{j,i} \) is type-definable is simply by the definability of types in stable theories, and in fact the action of \( \chi_{j,i} \) on the objects and morphisms of \( \mathcal{G}_j \) is given by the intersection of a definable set with the type-definable sets \( \text{Ob}(\mathcal{G}_j) \) and \( \text{Mor}(\mathcal{G}_j) \).

The formula (6) follows directly from the definition of \( \chi_{j,i}(f) \) in (2) and Lemma 2.13.

Next we prove (7). Suppose that \( i \leq j \leq k \). If \( a \in \text{Ob}(\mathcal{G}_k) \), then \( \chi_{j,i} \circ \chi_{k,j}(a) = \tau_{j,i}(\tau_{k,j}(a)) = \tau_{k,i}(a) = \chi_{k,i}(a) \). If \( a, b, c \in \text{Ob}(\mathcal{G}_k) \) and \( f = g \circ h \) are in (2) of the Lemma (with \( j \) replaced by \( k \)), then by the definition of the \( \chi \) maps,

\[
\chi_{j,i} \circ \chi_{k,j}(f) = \chi_{j,i}\left(\left[\pi_{k,j}\left(\langle h \rangle_k\right)\right]_i \circ \left[\pi_{k,j}\left(\langle g \rangle_k\right)\right]_i\right) \\
= \left[\pi_{j,i}\left(\left[\pi_{k,j}\left(\langle h \rangle_k\right)\right]_i\right)\right]_i \circ \left[\pi_{j,i}\left(\left[\pi_{k,j}\left(\langle g \rangle_k\right)\right]_i\right)\right]_i \\
= \left[\pi_{j,i}\left(\langle h \rangle_k\right)\right]_i \circ \left[\pi_{j,i}\left(\langle g \rangle_k\right)\right]_i = \chi_{j,i}(f).
\]

Finally, we check (8). Suppose \( i \leq j \) and \( f \in \text{Mor}_{\mathcal{G}_j}(a_{1}^{j}, a_{1}^{i}) \). To show that \( \psi_{i}(\chi_{j,i}(f)) = \rho_{j,i}(\psi_{j}(f)) \), we pick some arbitrary \( k_0 \in \text{Mor}_{\mathcal{G}_i}(a_{0}^{i}, a_{1}^{i}) \) and show that

\[
\psi_{i}(\chi_{j,i}(f))(k_0) = \rho_{j,i}(\psi_{j}(f))(k_0).
\]

On the one hand, by definition of \( \psi_{i} \),

\[
\psi_{i}(\chi_{j,i}(f))(k_0) = \chi_{j,i}(f) \circ k_0.
\]

To compute the right-hand side of equation 9, pick some \( k \in \text{Mor}_{\mathcal{G}_i}(a_{0}^{i}, a_{1}^{i}) \) such that \( \left[\pi_{j,i}(\langle k \rangle_j)\right]_i = k_0 \). Then

\[
[p_{j,i}(\psi_{j}(f))(k) = f \circ k,
\]
and \( \rho_{j,i}(\psi_{j}(f)) \) must move \( k_0 = \left[\pi_{j,i}(\langle k \rangle_j)\right]_i \) to the element which is defined from \( p_{j,i}(\langle f \circ k \rangle_j) \) in the same way that \( k_0 \) is defined from \( k \), so

\[
[p_{j,i}(\psi_{j}(f))(k_0) = \left[\pi_{j,i}(\langle f \circ k \rangle_j)\right]_i.
\]

By (6) and the functoriality of \( \chi_{j,i} \),

\[
[p_{j,i}(\psi_{j}(f))(k_0) = \chi_{j,i}(f \circ k) = \chi_{j,i}(f) \circ \chi_{j,i}(k) = \chi_{j,i}(f) \circ \left[\pi_{j,i}(\langle k \rangle_j)\right]_i = \chi_{j,i}(f) \circ k_0.
\]
So both sides of equation 9 equal \( \chi_{j,i}(f) \circ k_0 \), and we are done. \( \dashv \)
Finally, we define maps on the \( p \)-simplices and homology groups. Throughout, we will work with the set homology group (and set-simplices, et cetera) for convenience.

First, for every \( i \in I \), we pick an arbitrary “selection function” \( \alpha^0_i : S_0(p) \to p_i(C) \) such that \( \alpha^0_i(a) \in \text{dcl}(a) \). (This is a technical point, but the 0-simplices in \( S_0(p) \) are algebraic closures of realizations of \( p_i \), and there might be no canonical way to get a realization of \( p_i \) from a 0-simplex. Thus we need the choice functions \( \alpha^0_i \).

Next, we pick selection functions \( \alpha_i : S_1(p) \to \text{Mor}(\mathcal{G}_i) \) (for every \( i \in I \)) as follows. Suppose that \( \text{dom}(f) = \mathcal{P}(\{n_0, n_1\}) \) for \( n_0 < n_1 \), and for \( x \in \{n_0, n_1\} \), let “\( f_x \)” stand for \( f_{\{x\}} \alpha^0_i(f \upharpoonright \mathcal{P}(\{x\})) \) (remembering that things in the image of \( \alpha^0_i \) are realizations of \( p_i \), which are also objects in \( \text{Ob}(\mathcal{G}_i) \)). Then we pick \( \alpha_i(f) \) such that \( \alpha_i(f) \in \text{Mor}_{\mathcal{G}_i}(f_0, f_1) \). Just as in the proof of Lemma 2.10, we can use an inductive argument to ensure that if \( i \geq j \) then \( \chi_{ij}(\alpha_j(f)) = \alpha_i(f) \).

Finally, want to extend \( \alpha_i \) to a selection function \( \epsilon_i : S_2(p) \to \mathcal{G}_i \). To ease notation here and in what follows, we set the following notation:

**Notation 2.15.** Whenever \( f \in S_n(p) \), \( \text{dom}(f) = \mathcal{P}(s) \), and \( k \in s \), let

\[
 f^{k}_{k,s} := f_{s}^{\alpha^0_i(f \upharpoonright \mathcal{P}(\{k\}))},
\]

and note that \( f^{k}_{k,s} \) is a realization of \( p_i \), that is, an object in \( \mathcal{G}_i \). Similarly, if \( \{k, \ell\} \subseteq s \) and \( k < \ell \), let

\[
 f^{k,\ell}_{k,s} := f_{s}^{\alpha^0_i(f \upharpoonright \mathcal{P}(\{k, \ell\}))},
\]

which is a morphism in \( \text{Mor}_{\mathcal{G}_i}(f^{k}_{k,s}, f^{\ell}_{\ell,s}) \).

**Definition 2.16.** We define \( \epsilon_i : S_2(p) \to \mathcal{G}_i \) by the rule: if \( \text{dom}(f) = \mathcal{P}(s) \), where \( s = \{n_0, n_1, n_2\} \) and \( n_0 < n_1 < n_2 \), then we define \( \epsilon_i(f) \) as

\[
 \epsilon_i(f) := \left[ (f_{\{n_0, n_2\}, s}^{-1} \circ f_{\{n_1, n_2\}, s} \circ f_{\{n_0, n_1\}, s}) \right]_{\mathcal{G}_i}.
\]

(Recall that if \( f \in \text{Mor}_{\mathcal{G}_i}(a, a) \), then \( \lbrack f \rbrack_{\mathcal{G}_i} \) denotes the corresponding element of the group \( \mathcal{G}_i \).

These functions \( \epsilon_i \) can be extended linearly from \( S_2(p) \) to the collection of all 2-chains \( C_2(p) \), and by abuse of notation we also call this new function \( \epsilon_i \).

The next lemma is a technical point that will be useful for later computations.

**Lemma 2.17.** If \( i \in I \) and \( f \in S_n(p) \) for any \( n \geq 3 \), \( \text{dom}(g) = \mathcal{P}(t) \), and \( \{a, b, c\} \subseteq s \subseteq t \) with \( a < b < c \), then

\[
 \epsilon_i(f \upharpoonright \mathcal{P}(\{a, b, c\})) = \left[ (f_{\{a, c\}, s}^{-1} \circ f_{\{b, c\}, s} \circ f_{\{a, b\}, s}) \right]_{\mathcal{G}_i}.
\]
Proof. Remember that we identify the elements of $G_i$ with elements of acl($\emptyset$). Because the transition map $f_s^{(a,b,c)}$ fixes acl($\emptyset$) pointwise,

$$f_s^{(a,b,c)}(\epsilon_i(f \mid \{a, b, c\})) = \epsilon_i(f \mid \{a, b, c\}).$$

Therefore the left-hand side of the equation above equals $f_s^{(a,b,c)}(\epsilon_i(f \mid \{a, b, c\}))$, which is the equivalence class (in $G_i$) of

$$\left[ f_s^{(a,b,c)} \circ f_s^{(a,c)}(\alpha_i(f \mid P(\{a, c\}))) \right]^{-1} \circ$$

$$\left[ f_s^{(a,b,c)} \circ f_s^{(b,c)}(\alpha_i(f \mid P(\{b, c\}))) \circ \left[ f_s^{(a,b,c)} \circ f_s^{(a,b)}(\alpha_i(f \mid P(\{a, b\}))) \right] \right]^{-1} \circ$$

$$\left[ f_s^{(b,c)}(\alpha_i(f \mid P(\{b, c\}))) \right] \circ \left[ f_s^{(a,b)}(\alpha_i(f \mid P(\{a, b\}))) \right],$$

as desired. \qed

Lemma 2.18. If $c \in B_3^s(p)$, then for any $i \in I$, $\epsilon_i(c) = 0$.

Proof. By linearity, it suffices to check that $\epsilon_i(\partial(g)) = 0$ for any $g \in S_3^s(p)$. For simplicity of notation, we assume that dom$(g) = P(s)$ where $s = \{0, 1, 2, 3\}$. To further simplify, we write “$g_{i,j}$” for $g_{i(j,s)}$.

If $0 \leq j < k < \ell \leq 3$, by Lemma 2.17,

$$\epsilon_i(g \mid \{j, k, \ell\}) = \left[ g_{j,\ell}^{-1} \circ g_{k,\ell} \circ g_{j,k} \right]_{G_i}.$$

Therefore $\epsilon_i(\partial(g))$ equals

$$\left[ g_{1,3}^{-1} \circ g_{2,3} \circ g_{1,2} \right]_{G_i} - \left[ g_{0,3}^{-1} \circ g_{2,3} \circ g_{0,2} \right]_{G_i}$$

$$+ \left[ g_{0,3}^{-1} \circ g_{1,3} \circ g_{0,1} \right]_{G_i} - \left[ g_{0,2}^{-1} \circ g_{1,2} \circ g_{0,1} \right]_{G_i}$$

$$= \left[ g_{0,1}^{-1} \circ g_{1,3}^{-1} \circ g_{2,3} \circ g_{1,2} \circ g_{0,1} \right]_{G_i} - \left[ g_{0,3}^{-1} \circ g_{2,3} \circ g_{0,2} \right]_{G_i}$$

$$+ \left[ g_{0,3}^{-1} \circ g_{1,3} \circ g_{0,1} \right]_{G_i} - \left[ g_{0,2}^{-1} \circ g_{1,2} \circ g_{0,1} \right]_{G_i}$$

$$= - \left[ g_{0,1}^{-1} \circ g_{1,2} \circ g_{0,1} \right]_{G_i} - \left[ g_{0,3}^{-1} \circ g_{2,3} \circ g_{0,2} \right]_{G_i} + \left[ g_{0,3}^{-1} \circ g_{1,3} \circ g_{0,1} \right]_{G_i}$$

$$+ \left[ g_{0,1}^{-1} \circ g_{1,3} \circ g_{2,3} \circ g_{1,2} \circ g_{0,1} \right]_{G_i}$$

but everything in the last expression cancels out. \qed
By the last lemma, each $\epsilon_i$ induces a well-defined function $\overline{\epsilon}_i : H_2(p) \to G_i$.

Now we relate the $\epsilon_i$ maps to the groupoid maps $\chi_{j,i} : G_j \to G_i$. For $i \in I$, let $\overline{\psi}_i : G_i \to \text{Aut}(SW_i/\overline{a_0}, \overline{\pi})$ be the map induced by $\psi_i : \text{Mor}_{G_i}(a_i^1, a_i^1) \to \text{Aut}(SW_i/\overline{a_0}, \overline{\pi})$, and let $\overline{\chi}_{j,i} : G_j \to G_i$ be the surjective group homomorphism induced by the functor $\chi_{j,i}$ from Lemma 2.14.

Everything coheres:

**Lemma 2.19.** If $i \leq j \in I$ and $f \in S_2(p)$, then $\overline{\chi}_{j,i}(\epsilon_j(f)) = \epsilon_i(f)$.

**Proof.** For convenience, we assume that $\text{dom}(f) = [2] = \{0, 1, 2\}$. Also, in the proof of this lemma, we write $"f^i_{k,\ell}"$ for $"f^i_{[k,\ell],[2]}"$ (as in Notation 2.15).

**Claim 2.20.** If $i \leq j$, then $\chi_{j,i}(f^j_{k,\ell}) = f^i_{k,\ell}$.

**Proof.** The left-hand side is, by definition, equal to

$$
\chi_{j,i} \left( f^j_{[2]}(\alpha_j(f \restriction \{k, \ell\})) \right) = \left[ \pi_{j,i} \left( f^j_{[2]}(\alpha_j(f \restriction \{k, \ell\})) \right) \right]_i
$$

(using (6) of Lemma 2.14). But the map $f^j_{[2]}$ is elementary and the functions $\pi_{j,i}, \langle \cdot \rangle_j$, and $[\cdot]_i$ are all definable, so this expression equals

$$
f^j_{[2]} \left( \pi_{j,i}(\langle \alpha_j(f \restriction \{k, \ell\}) \rangle_j) \right) = f^j_{[2]}(\chi_{j,i}(\alpha_j(f \restriction \{k, \ell\})))
$$

$$
\quad = f^i_{[2]}(\alpha_i(f \restriction \{k, \ell\})),
$$

by our choice of the $\alpha_i$ functions such that $\chi_{j,i} \circ \alpha_j = \alpha_i$. But this last expression equals the right-hand side in the Claim. $\dashv$

To prove the lemma, first pick some (any) morphism $g \in \text{Mor}_{G_j}(a_j^1, f_0)$, and note that $\epsilon_j(f)$ is an element of the group $G_j$ which is represented by the following morphism in $\text{Mor}_{G_j}(a_j^1, a_j^1)$:

$$
g^{-1} \circ (f^j_{0,2})^{-1} \circ f^j_{1,2} \circ f^j_{0,1} \circ g.
$$

So $\overline{\chi}_{j,i}(\epsilon_j(f))$ is represented by the morphism

$$
\chi_{j,i} \left( g^{-1} \circ (f^j_{0,2})^{-1} \circ f^j_{1,2} \circ f^j_{0,1} \circ g \right)
$$

$$
= \chi_{j,i}(g)^{-1} \circ \chi_{j,i}(f^j_{0,2})^{-1} \circ \chi_{j,i}(f^j_{1,2}) \circ \chi_{j,i}(f^j_{0,1}) \circ \chi_{j,i}(g),
$$

which, by the Claim above, equals

$$
\chi_{j,i}(g)^{-1} \circ (f^j_{0,2})^{-1} \circ f^j_{1,2} \circ f^j_{0,1} \circ \chi_{j,i}(g),
$$

which, by definition, is a representative of $\epsilon_i(f)$. $\dashv$
Let $G$ be the limit of the inverse system of groups $\langle G_i : i \in I \rangle$ with transition maps given by the $\chi_{j,i} : G_j \to G_i$. By Lemma 2.19, the maps $\tilde{\epsilon}_i$ induce a group homomorphism $\epsilon : H_2(p) \to G$.

**Lemma 2.21.** The map $\epsilon : H_2(p) \to G$ is injective. In other words, if $c \in Z_2(p)$ and $\epsilon_i(c) = 0$ for every $i \in I$, then $c \in B_2(p)$.

**Proof.** Since $Z_2(p)$ is generated over $B_2(p)$ by all the 2-shells, it is enough to prove this in the case where $c$ is a 2-shell of the form $f_0 - f_1 + f_2 - f_3$, where $f_0$ is a 2-simplex with domain $\mathcal{P}([3] \setminus \{a\})$. We will construct a 3-simplex $g : \mathcal{P}([3]) \to \mathcal{C}$ such that $\partial(g) = c$.

Pick some $a_3 = p|(a_0, a_1, a_2)$, so that $(a_0, a_1, a_2, a_3) = p^{(4)}$. We will construct $g$ so that $g([3]) = \overline{\mathcal{a}_3}$. If $(b, c, d, e)$ is some permutation of $(0, 1, 2, 3)$, then $f_{b,c,d,e}(\{b, c\}) = f_{b,c,d,e}(\{b, c\})$ (since $\partial(c) = 0$), and we can assume that $f_{b,c,d,e}(\{b, c\}) = \overline{\mathcal{a}_3} = f_{b,c,d,e}(\{b, c\})$.

As a first step in defining the simplex $g$, for any $\{b, c\} \subseteq \{0, 1, 2, 3\}$, we let $\hat{g} | \{b, c\} = f | \{b, c, d\}$ (where $d$ is any other element of $[3]$), and we let the maps $g_{b,c,d}^b : \overline{\mathcal{a}_3} \to \overline{\mathcal{a}_3}$ be the inclusion maps. We take the transition map $g_{b,c}^b$ (for $b \in [3]$) to be the identity map from $\overline{\mathcal{a}_3}$ to itself.

Next we will define the transition maps $g_{b,c}^b : \overline{\mathcal{a}_3} \to \overline{\mathcal{a}_3}$ in such a way as to ensure compatibility with the faces $f_b$. Before doing this, we set some notation. First, we write “$f_{x,y,z}^i$” for the set $(f_z)^{(1)} \{x, y, \{3\} \setminus \{z\}\}$ as in Notation 2.15. Similarly, we write

$$f_{b,c,d} := (f_d)^{(b,c)}(\overline{\mathcal{a}_3}).$$

We consider the sets $\overline{\mathcal{a}_3}$ to be 1-simplices in which all of the transition maps are inclusions and the “vertices” are $\overline{\mathcal{a}_6}$ and $\overline{\mathcal{a}_3}$. This allows us to write “$\alpha_i(\overline{\mathcal{a}_6})$.” For $i \in I$ and $\{b, c\} \subseteq [3]$, let $e_i$ be the “edge” $\alpha_i(\overline{\mathcal{a}_c})$.

We define the maps $g_{b,c}^0$, $g_{b,c}^1$, and $g_{b,c}^2$ to be the identity maps. Then we define the other three edge transition maps $g_{b,c}^i$, $g_{b,c}^i$, and $g_{b,c}^i$ so that for every $i \in I$,

\begin{align*}
(10) & \quad g_{b,c}^1(e_{13}) g_{b,c}^2(e_{23}) g_{b,c}^2(e_{12}) \equiv_{\text{acl}(0)} f_{13,0}^i f_{23,0}^i f_{12,0}^i, \\
(11) & \quad g_{b,c}^0(e_{03}) g_{b,c}^2(e_{23}) g_{b,c}^2(e_{02}) \equiv_{\text{acl}(0)} f_{03,1}^i f_{23,1}^i f_{02,1}^i, \\
& \quad \text{and} \\
(12) & \quad g_{b,c}^0(e_{03}) g_{b,c}^1(e_{13}) g_{b,c}^0(e_{01}) \equiv_{\text{acl}(0)} f_{03,2}^i f_{13,2}^i f_{01,2}^i.
\end{align*}
Having specified values according to the three equations above, we let \( g^{01}_{[3]} \), \( g^{12}_{[3]} \), and \( g^{02}_{[3]} \) be any elementary extensions to the respective domains \( a_{03}, a_{13}, \) and \( a_{23} \).

**Claim 2.22.** For any \( i \in I \),

\[
(13) \quad g^{02}_{[3]}(e_{02}^i) g^{12}_{[3]}(e_{12}^i) g^{01}_{[3]}(e_{01}^i) = f^i_{02,3} f^i_{12,3} f^i_{01,3}.
\]

**Proof.** Note that by stationarity,

\[
g^{02}_{[3]}(e_{02}^i) g^{12}_{[3]}(e_{12}^i) \equiv f^i_{02,3} f^i_{12,3},
\]

and to check the Claim, it suffices to show that

\[
[g^{02}_{[3]}(e_{02}^i)]^{-1} \circ g^{12}_{[3]}(e_{12}^i) \circ g^{01}_{[3]}(e_{01}^i) \equiv (f^i_{02,3})^{-1} \circ f^i_{12,3} \circ f^i_{01,3}.
\]

The right-hand side equals \( \epsilon_i(f_3) \). Since \( \epsilon_i(c) = 0 \),

\[
\epsilon_i(f_3) = \epsilon_i(f_0) - \epsilon_i(f_1) + \epsilon_i(f_2).
\]

Let \( "g_{bc}" \) be an abbreviation for \( g^{bc}_{[3]}(e_{bc}^i) \). By applying equations 10, 11, and 12 above (and performing a very similar calculation as in the proof of Lemma 2.18), we get:

\[
\epsilon_i(f_3) = [g^{13}_{[3]} \circ g_{23} \circ g_{12}]_{G_i} - [g^{03}_{[3]} \circ g_{23} \circ g_{02}]_{G_i} + [g^{03}_{[3]} \circ g_{13} \circ g_{01}]_{G_i},
\]

\[
= [g^{03}_{[3]} \circ g_{23}]_{G_i} + [g^{03}_{[3]} \circ g_{13}]_{G_i} + [g^{01}_{[3]} \circ g_{01}]_{G_i} - [g^{03}_{[3]} \circ g_{23}]_{G_i} + [g^{03}_{[3]} \circ g_{13}]_{G_i} + [g^{01}_{[3]} \circ g_{01}]_{G_i},
\]

\[
= [g^{03}_{[3]} \circ g_{23}]_{G_i} + [g^{03}_{[3]} \circ g_{13}]_{G_i} + [g^{01}_{[3]} \circ g_{01}]_{G_i} - [g^{03}_{[3]} \circ g_{23}]_{G_i} + [g^{03}_{[3]} \circ g_{13}]_{G_i} + [g^{01}_{[3]} \circ g_{01}]_{G_i},
\]

\[
= [g^{03}_{[3]} \circ g_{23}]_{G_i} + [g^{03}_{[3]} \circ g_{13}]_{G_i} + [g^{01}_{[3]} \circ g_{01}]_{G_i} - [g^{03}_{[3]} \circ g_{23}]_{G_i} + [g^{03}_{[3]} \circ g_{13}]_{G_i} + [g^{01}_{[3]} \circ g_{01}]_{G_i},
\]

\[
= [g^{03}_{[3]} \circ g_{23}]_{G_i} + [g^{03}_{[3]} \circ g_{13}]_{G_i} + [g^{01}_{[3]} \circ g_{01}]_{G_i},
\]

as desired.

Now we must check that this coheres with the types of the given simplices \( f_5 \):

**Claim 2.23.** If \( \langle b, c, d, e \rangle \) is a permutation of \( \{3\} \) with \( 0 \leq b < c < d \leq 3 \), then

\[
g^{bd}_{[3]}(\overline{a_{bd}}) g^{cd}_{[3]}(\overline{a_{cd}}) g^{bc}_{[3]}(\overline{a_{bc}}) \equiv f_{bd, \overline{e}} f_{cd, \overline{e}} f_{bd, \overline{e}}.
\]
Proof. Let

\[ \tilde{f}_{xy,\bar{c}} := \bigcup_{i \in I} f_{xy,\bar{c}}^i. \]

Then Claim 2.23 follows from Claim 2.22 above together with:

**Subclaim 2.24.** If \((x, y, z)\) is any permutation of \((b, c, d)\) with \(x < y\), then \(\text{tp}(f_{xy,\bar{c}}/f_{yz,\bar{c}}f_{zx,\bar{c}})\) is isolated by \(\text{tp}(f_{xy,\bar{c}}/f_{yz,\bar{c}}f_{zx,\bar{c}})\).

Proof. Note that \(f_{xy,\bar{c}} \subseteq \text{acl}(f_{yz,\bar{c}}, f_{zx,\bar{c}})\) (in fact, it is in the algebraic closure of the “vertices” \(f_{x,\bar{c}} \subseteq f_{xz,\bar{c}}\) and \(f_{y,\bar{c}} \subseteq f_{yz,\bar{c}}\)). Suppose towards a contradiction that \(h \in f_{xy,\bar{c}}\) but

\[ \text{tp}(h/f_{yz,\bar{c}}f_{zx,\bar{c}}) \not\models \text{tp}(h/f_{yz,\bar{c}}f_{zx,\bar{c}}). \]

This means that the orbit of \(h\) under \(\text{Aut}(\mathcal{C}/f_{yz,\bar{c}}f_{zx,\bar{c}})\) is smaller than the orbit of \(h\) under \(\text{Aut}(\mathcal{C}/f_{yz,\bar{c}}f_{zx,\bar{c}})\). Let \(\hat{h}\) be a name for the orbit of \(h\) under \(\text{Aut}(\mathcal{C}/f_{yz,\bar{c}}f_{zx,\bar{c}})\) as a set. Then

\[ \hat{h} \in \text{dcl}(f_{yz,\bar{c}}, f_{zx,\bar{c}}) \setminus \text{dcl}(f_{yz,\bar{c}}, f_{zx,\bar{c}}). \]

Since \(\hat{h} \in \text{dcl}(f_{yz,\bar{c}}f_{zx,\bar{c}})\), it lies in \(f_{xy,\bar{c}}^i\) for some \(i \in I\) (this is by the maximality condition on our symmetric witnesses \(\langle W_i : i \in I \rangle\)). Also,

\[ f_{xy,\bar{c}}^i \subseteq \text{dcl}(f_{yz,\bar{c}}, f_{zx,\bar{c}}) \]

due to the fact that \(f_{xy,\bar{c}}^i\) is interdefinable with the set of all morphisms in \(\text{Mor}_{\mathcal{G}_i}(f_{x,\bar{c}}, f_{y,\bar{c}})\), which can be obtained via composition in \(\mathcal{G}_i\) from the corresponding morphisms in \(\text{dcl}(f_{yz,\bar{c}})\) and \(\text{dcl}(f_{zx,\bar{c}})\). But this contradicts the fact that \(\hat{h} \notin \text{dcl}(f_{yz,\bar{c}}f_{zx,\bar{c}})\).

\[ \neg \]

Claim 2.23 implies that for each permutation \((b, c, d, e)\) of \([3]\), we can find an elementary map \(g^{b,d,c}_{[3]}\) from the “face” \(f_2([3] \setminus \{e\})\) onto \(a_{b,c,d,e}\) which is coherent with the maps \(g^{b,c}_{[3]}, g^{c,d}_{[3]}\), and \(g^{b,d}_{[3]}\) that we have already defined, and such that \(\partial^g f = f_i\). This completes the proof of Lemma 2.21.

\[ \neg \]

**Lemma 2.25.** The map \(\epsilon : H_2(p) \to G\) is surjective.

Proof. Suppose that \(g\) is any element in \(G\), and that \(g\) is represented by a sequence \(\langle g_i : i \in I \rangle\) such that \(\sum_{j \leq i} g_j = g_i\) whenever \(i \leq j\). We will construct a 2-chain \(c = f - h\) such that \(\epsilon_i(f - h) = g_i\) for every \(i \in I\), which will establish the Lemma. Let \(f : \mathcal{P}([2]) \to \mathcal{C}\) be the 2-simplex such that \(f(s) = \overline{a_s}\) for every \(s \subseteq [2]\) and such that every transition map in \(f\) is an inclusion map. Let \(k_i = \epsilon_i(f)\).
We want to construct \( h : \mathcal{P}([2]) \to \mathfrak{C} \) such that \( h([2]) = \varpi_{(0)} \), \( \partial(h) = \partial(f) \), and \( h_{[2]}^i \) is the identity map whenever \( s \subseteq \{0, 1\} \) or \( s \subseteq \{1, 2\} \). The only thing left is to specify an elementary map \( h_{[2]}^{02} : \varpi_0 \to \varpi_0 \) fixing \( \varpi_0 \) and \( \varpi_2 \).

**Claim 2.26.** Suppose that \( h_i \in \text{Mor}_{\mathcal{G}_i}(a_0^i, a_2^i) \) is the unique element such that \( [h_i^{-1} \circ h_{12}^i \circ h_{01}^i]_{\mathcal{G}_i} = k_i - g_i \). Then

1. whenever \( i \leq j \), \( \chi_{j,i}(h_j) = h_i \), and
2. \( \text{tp}(h_0, \ldots, h_i/\varpi_0, \varpi_2) = \text{tp}(\alpha_0(\varpi_0), \ldots, \alpha_i(\varpi_0)/\varpi_0, \varpi_2) \).

**Proof.** First we show:

**Subclaim 2.27.** \( \chi_{i+1,i}(h_{12}^{i+1}) = h_{12}^i \) and \( \chi_{i+1,i}(h_{01}^{i+1}) = h_{01}^i \).

**Proof.** We check only the first equation (and the second equation has an identical proof). By (6) of Lemma 2.14,

\[
\chi_{i+1,i}(h_{12}^{i+1}) = [\pi_{i+1,i}((h_{12}^{i+1})_{i+1})]_i = [\pi_{i+1,i}((h_{12}^{i+1}(\alpha_{i+1}(\varpi_{12})))_{i+1})]_i
\]

\[
= h_{[2]}^{12}([\pi_{i+1,i}((\alpha_{i+1}(\varpi_{12}))_{i+1})]_i) = h_{[2]}^{12}(\chi_{i+1,i}(\alpha_{i+1}(\varpi_{12})))
\]

\[
= h_{[2]}^{12}(\alpha_i(\varpi_{12})) = h_i^{12}.
\]

Note that it is enough to prove (1) of the Claim for every pair \((i, j)\) where \( j = i + 1 \). We apply \( \chi_{i+1,i} \) to both sides of the equation \( [h_i^{-1} \circ h_{12}^i \circ h_{01}^i]_{\mathcal{G}_{i+1}} = k_{i+1} - g_{i+1} \). On the right-hand side, this yields

\[
\chi_{i+1,i}(k_{i+1} - g_{i+1}) = k_i - g_i.
\]

On the left-hand side, using the Subclaim, we get

\[
[\chi_{i+1,i}(h_i^{-1} \circ h_{12}^i \circ h_{01}^i)]_{\mathcal{G}_i} = [\chi_{i+1,i}(h_{i+1})^{-1} \circ h_{12}^i \circ h_{01}^i]_{\mathcal{G}_i}.
\]

So putting together Equations 14 and 15, we get that

\[
[\chi_{i+1,i}(h_{i+1})^{-1} \circ h_{12}^i \circ h_{01}^i]_{\mathcal{G}_i} = k_i - g_i.
\]

But \( h_i \) is the unique morphism in \( \mathcal{G}_i \) such that \( [h_i^{-1} \circ h_{12}^i \circ h_{01}^i]_{\mathcal{G}_i} = k_i - g_i \), so part (1) of the Claim follows.

We prove part (2) by induction on \( i \in I \). The base case follows from

\[
\text{tp}(h_0/\varpi_0, \varpi_2) = \text{tp}(\alpha_0(\varpi_0)/\varpi_0, \varpi_2)
\]
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(which is true simply because both elements belong to \( SW(a^0, a^1) \)). If (2) is true for \( i \), then to prove it for \( i+1 \), it is enough to check that

\[
\text{tp}(\langle h_{i+1} \rangle_{i+1}, \langle a \rangle_{i+1}) = \text{tp}(\langle \alpha_{i+1}(a^0) \rangle_{i+1}, \langle \alpha_i(a^1) \rangle_{i+1}),
\]

since all the other elements are in the definable closure of \( h_i \) and \( \alpha_i(a^1) \) via the maps \( \chi_{k,\ell} \). To see this, first note that

\[
\text{tp}(\langle h_{i+1} \rangle_{i+1}/a_0, a_2) = \text{tp}(\langle \alpha_{i+1}(a^0) \rangle_{i+1}/a_0, a_2)
\]

just because both elements belong to \( SW(a^0, a^1) \). By part (1) of the Claim, \( \pi_{i+1,i}(\langle h_{i+1} \rangle_{i+1}) = \langle h_i \rangle_i \), and by the way we chose the \( \alpha \) functions, \( \pi_{i+1,i}(\langle \alpha_{i+1}(a^0) \rangle_{i+1}) = \langle \alpha_i(a^1) \rangle_i \). Since the function \( \pi_{i+1,i} \) is definable, Equation 16 follows.

Given the elements \( h_i \) as in the Claim above, we let \( h^0_2 : a^0 \to a^0 \)
be any elementary map that fixes \( a_0 \cup a_2 \) pointwise and maps each element \( \alpha_i(a^0) \) to \( h_i \). Then \( \epsilon_i(f - h) = k_i - (k_i - g_i) = g_i \), as desired.

By Lemmas 2.21 and 2.25, \( H_2(p) \cong G \). To finish the proof of Theorem 2.1, we just need to show:

**Lemma 2.28.** \( G \cong \text{Aut}(a^0a_1/a_0, a_1) \)

**Proof.** Note that \( \text{Aut}(a^0a_1/a_0, a_1) \) is the limit of the groups \( \text{Aut}(SW_i/a_0, a_1) \)
via the transition maps \( \rho_{j,i} : \text{Aut}(SW_j/a_0, a_1) \to \text{Aut}(SW_i/a_0, a_1) \), due
to the maximality condition that every element of \( a^0a_1 \) lies in one of the symmetric witnesses \( SW_i \). Also, part (8) of Lemma 2.14 implies that we have a commuting system

\[
\begin{array}{ccc}
G_j & \xrightarrow{\psi_j} & G_i \\
\downarrow & & \downarrow \\
\text{Aut}(SW_j/a_0, a_1) & \xrightarrow{\rho_{j,i}} & \text{Aut}(SW_i/a_0, a_1)
\end{array}
\]

But the maps \( \psi_i \) are all isomorphisms, so taking limits we get an
isomorphism from \( G \) to \( \text{Aut}(a^0a_1/a_0, a_1) \).

3. **ANY PROFINITE ABELIAN GROUP CAN OCCUR AS \( H_2(p) \)**

In this section, we construct a family of examples which prove the following:

**Theorem 3.1.** For any profinite abelian group \( G \), there is a type \( p \) in
a stable theory \( T \) such that \( H_2(p) \cong G \). In fact, we can build the theory
\( T \) to be totally categorical.
Together with Theorem 2.1 from the previous section, this shows that the groups that can occur as $H_2(p)$ for a type $p$ in a stable theory are precisely the profinite abelian groups.

For the remainder of this section, we fix a profinite abelian group $G$ which is the inverse limit of the system $\langle H_i : i \in I \rangle$, where each $H_i$ is finite and abelian, $(I, \leq)$ is a directed set, and $G$ is the limit along the surjective group homomorphisms $\varphi_{j,i} : H_j \to H_i$ (for every pair $i \leq j$ in $I$). The language $L$ of $T$ will be as follows: there will be a sort $G_i$ for each $i \in I$, and function symbols $\chi_{j,i} : G_j \to G_i$ for every pair $i \leq j$. The theory $T$ will say, in the usual language of categories, that each $G_i$ is a connected groupoid with infinitely many objects, and there will be separate composition symbols for each sort $G_i$. Also, $T$ says that $G_i$ is a groupoid such that each vertex group $\text{Mor}_{G_i}(a_i, a_i)$ is isomorphic to the group $H_i$. For convenience, pick some arbitrary $a_i \in \text{Ob}(G_i)$ and some group isomorphism $\xi_i : G_i \to \text{Mor}_{G_i}(a_i, a_i)$ (but the $\xi_i$'s are not a part of any model of $T$). Then the last requirement we make on $T$ is that the function symbols $\chi_{j,i}$ define full functors from $G_j$ onto $G_i$ which induce bijections between the corresponding collections of objects, and such that for every pair $i \leq j$, the following diagram commutes:

$$
\begin{array}{ccc}
H_j & \xrightarrow{\varphi_{j,i}} & H_i \\
\downarrow{\xi_i} & & \downarrow{\xi_i} \\
\text{Mor}_{G_j}(a_j, a_j) & \xrightarrow{\chi_{j,i}} & \text{Mor}_{G_i}(a_i, a_i)
\end{array}
$$

(In other words, the functors $\chi_{j,i}$ are just “isomorphic copies” the group homomorphisms $\varphi_{j,i}$.)

**Lemma 3.2.** The theory $T$ described above is complete and admits elimination of quantifiers. If we further assume that the language is multi-sorted and that every element of a model must belong to one of the sorts $G_i$, then $T$ is totally categorical.

**Proof.** If the language is multi-sorted, then since the groupoids $G_i$ are all connected and there are bijections between the object sets of the various $G_i$, the isomorphism class of a model of $T$ is determined by the cardinality of the object set of some (any) $G_i$. This shows that $T$ is totally categorical, hence $T$ is complete.

For quantifier elimination, it suffices to show the following: for any two models $M_1$ and $M_2$ of $T$ with a common substructure $A$ and any sentence $\sigma$ with parameters from $A$ of the form $\sigma = \exists x \varphi(x; \overline{a})$ where $\varphi$ is quantifier-free, if $M_1 \models \sigma$, then $M_2 \models \sigma$. (See Theorem 8.5 of [14].) In this situation, let $\text{cl}(A)$ denote the submodel of $M_1$ (and of
$M_2$) generated by $A$, and in case $A = \emptyset$, let $\text{cl}(A) = \emptyset$. Then if $M_1 \models \sigma$ as above, at least one of the following is true:

1. $\varphi(x; \bar{a})$ is satisfied by some $x$ in $\text{cl}(A)$;
2. $\varphi(x; \bar{a})$ is satisfied by some morphism between two objects in $\text{cl}(A)$;
3. For some $i \in I$, $\varphi(x; \bar{a})$ is satisfied by any object in $G_i$ outside of $\text{cl}(A)$;
4. For some $i \in I$, $\varphi(x; \bar{a})$ is satisfied by any morphism in $G_i$ whose source and target are both outside of $\text{cl}(A)$.

In each of the five cases above, it is straightforward to check that there is an $x$ realizing $\varphi(x; \bar{a})$ in $M_2$ as well (for the last three cases we use the fact that $\text{Ob}(G_i)$ is infinite).

\begin{remark}
If $A \subseteq G_i$, then we say that $b \in \text{Ob}(G_i)$ is connected to $A$ if either $b \in A$ or $b$ is the source or target of a morphism in $A$. By elimination of quantifiers, it follows that for any $A \subseteq G_i$, $\text{acl}(A) \cap G_i$ is the union of all objects $b$ that are connected to $A$ plus all morphisms $f \in \text{Mor}_{G_i}(b, c)$ such that $b$ and $c$ are connected to $A$.

Because of the functors $\chi_{j,i}$, it follows that for any $a$ in any $G_i$, $\text{acl}(a)$ actually contains objects and morphisms from each of the groupoids $G_j$. But for any $A \subseteq \mathfrak{C}$, we can write $\text{acl}(A)$ in the “standard form” $\text{acl}(A) = \text{acl}(A_0)$ for some $A_0 \subseteq \text{Ob}(G_0)$, and:

1. $\text{acl}(A_0) \cap \text{Ob}(G_i) = \chi_{i,0}^{-1}(A_0)$, and
2. $\text{acl}(A_0) \cap \text{Mor}(G_i)$ is the collection of all $f \in \text{Mor}_{G_i}(b, c)$ where $b, c \in \text{acl}(A_0)$.

\end{remark}

\begin{lemma}
The theory $T$ has weak elimination of imaginaries in the sense of [13]: for every formula $\varphi(\bar{x}, \bar{a})$ defined over a model $M$ of $T$, there is a smallest algebraically closed set $A \subseteq M$ such that $\varphi(\bar{x}, \bar{a})$ is equivalent to a formula with parameters in $A$.

\begin{proof}
By Lemma 16.17 of [13], it suffices to prove the following two statements:

1. There is no strictly decreasing sequence $A_0 \supseteq A_1 \supseteq \ldots$, where every $A_i$ is the algebraic closure of a finite set of parameters; and
2. If $A$ and $B$ are algebraic closures of finite sets of parameters in the monster model $\mathfrak{C}$, then $\text{Aut}(\mathfrak{C} / A \cap B)$ is generated by $\text{Aut}(\mathfrak{C} / A)$ and $\text{Aut}(\mathfrak{C} / B)$.
\end{proof}

Statement 1 follows immediately from the characterization of algebraically closed sets in Remark 3.3 above (that is, algebraic closures of finite sets are equivalent to algebraic closures of finite subsets of $\text{Ob}(G_0)$).

To check statement 2, suppose that $\sigma \in \text{Aut}(\mathcal{C}/A \cap B)$, and assume that $A = \text{acl}(A_0)$ and $B = \text{acl}(B_0)$ where $A_0, B_0 \subseteq \text{Ob}(G_0)$. Note that any permutation of $\text{Ob}(G_0)/A_0 \cap B_0$ which fixes $A_0$ can be extended to an automorphism of $\text{Aut}(\mathcal{C}/A)$, and likewise for $B_0$ and $B$. So as a first step, we can use the fact that $\text{Sym}(\text{Ob}(G_0)/A_0 \cap B_0)$ is generated by $\text{Sym}(\text{Ob}(G_0)/A_0)$ and $\text{Sym}(\text{Ob}(G_0)/B_0)$ to find an automorphism $\tau \in \text{Aut}(\mathcal{C})$ such that $\tau$ is in the subgroup generated by $\text{Aut}(\mathcal{C}/A)$ and $\text{Aut}(\mathcal{C}/B)$ and $\sigma \circ \tau^{-1}$ fixes $\text{Ob}(G_0)$ (and hence $\text{Ob}(G_i)$ for every $i$) pointwise.

Finally, we need to deal with the morphisms. We claim that there is a map $\sigma^0_A \in \text{Aut}(\mathcal{C}/A)$ which fixes $\text{Ob}(G_0)$ pointwise and such that for any $f \in \text{Mor}_{G_0}(b,c)$ such that at least one of $b$ and $c$ do not lie in $A$, $(\sigma^0_A \circ \tau)(f) = \sigma(f)$. (The idea is to use the recipe for constructing object-fixing automorphisms described in subsection 4.2 of [4], using a basepoint $a_0 \in A$.) In fact, by the same argument we can also assume that for every $i \in I$ and for any $f \in \text{Mor}_{G_i}(b,c)$ such that at least one of $b$ and $c$ do not lie in $A$, $(\sigma^0_A \circ \tau)(f) = \sigma(f)$. Similarly, there is a map $\sigma^0_B \in \text{Aut}(\mathcal{C}/B)$ which fixes $\text{Ob}(G_0)$ pointwise and for any $i \in I$, $\sigma^0_B$ only moves morphisms in $\text{Mor}_{G_i}(b,c)$ where $b$ and $c$ are both in $A \setminus (A \cap B)$, and such that $\sigma^0_B \circ \sigma^0_A \circ \tau = \sigma$.

\[\square\]

**Lemma 3.5.** If $a^0, a^1 \in \text{Ob}(G_i)$, then

\[
\text{acl}^{eq}(a^0, a^1) = \text{dcl}^{eq}\left( \bigcup_{i,j \in I, i \leq j} \text{Mor}_{G_j}(a^0_j, a^1_j) \right),
\]

where $a^i_j = \chi_{j,i}^{-1}(a^i)$.

**Proof.** Suppose $g \in \text{acl}^{eq}(a^0, a^1)$. Then $g = b/E$ for some $(a^0, a^1)$-definable finite equivalence relation $E$. By Lemma 3.4, there is a finite tuple $\bar{d} \in \mathcal{C}$ (in the home sort) such that $b/E$ is definable over $\bar{d}$ and $\bar{d}$ has a minimal algebraic closure. If the set $\text{acl}(\bar{d})$ contained an object $a$ of $G_0$ other than $\chi_{i,0}(a^0)$ and $\chi_{j,0}(a^1)$, then (by quantifier elimination) $\text{acl}(\bar{d})$ would have an infinite orbit under $\text{Aut}(\mathcal{C}/a^0, a^1)$, and so $E$ would have infinitely many classes, a contradiction. So by Remark 3.3, the set $\bar{d}$, and hence $b/E$ is definable over the union of the morphism sets $\text{Mor}_{G_j}(a^0_j, a^1_j)$.

\[\square\]
From now on, we assume that all algebraic and definable closures are computed in $T^e$, not just in the home sort.

**Lemma 3.6.** If $a^0, a^1 \in \text{Ob}(\mathcal{G}_i)$, then for any two $f, g \in \text{Mor}_\mathcal{G}_i(a^0, a^1)$, 
\[
\text{tp}(f/\text{acl}(a^0), \text{acl}(a^1)) = \text{tp}(g/\text{acl}(a^0), \text{acl}(a^1)).
\]

**Proof.** Using the same procedure as described in subsection 4.2 of [4], we can construct an automorphism $\sigma$ of $\mathcal{C}$ fixing $\text{Ob}(\mathcal{G}_i)$, $\text{Mor}_\mathcal{G}_i(a^0, a^0)$, and $\text{Mor}_\mathcal{G}_i(a^1, a^1)$ pointwise while mapping $f$ to $g$. (In the construction of [4], the “basepoint” $a_0$ there can be chosen to be $a^0$ here, and then condition (5) of the construction plus the fact that $\mathcal{G}_i$ is abelian implies that $\text{Mor}_\mathcal{G}_i(a^1, a^1)$ is fixed.) In fact, it is easy to see that we can even ensure that $\sigma$ fixes $\text{Mor}_\mathcal{G}_i(\chi_{j,i}^{-1}(a^0), \chi_{j,i}^{-1}(a^0))$ and $\text{Mor}_\mathcal{G}_i(\chi_{j,i}^{-1}(a^1), \chi_{j,i}^{-1}(a^1))$ pointwise, so by Lemma 3.5, $\sigma$ fixes $\text{acl}(a^0) \cup \text{acl}(a^1)$ pointwise. \hfill \qed

Let $p = \text{stp}(a_0)$ for some (any) $a_0 \in \text{Ob}(\mathcal{G}_0)$.

**Proposition 3.7.** $H_2(p) \cong G$.

**Proof.** Pick $(a^0, a^1, a^2) \models p^{(3)}$. By Theorem 2.1 (the “Hurewicz theorem”), it is enough to show that $\text{Aut}(\overline{a^0a^1}/\overline{a^0, a^1}) \cong G$. For ease of notation, let $a^k_i = \chi_{i,0}^{-1}(a^k)$ for $k = 0, 1$, or $2$. By Lemma 3.5 and the fact that any morphism in $\text{Mor}_\mathcal{G}_i(a^0_i, a^1_i)$ is a composition of morphisms in $\text{Mor}_\mathcal{G}_i(a^0, a^2)$ and $\text{Mor}_\mathcal{G}_i(a^2_i, a_i^1)$, it follows that the set $\overline{a^0a^1}$ is interdefinable with $\bigcup_{i \in I} \text{Mor}_\mathcal{G}_i(a^0_i, a^1_i)$.

So $\text{Aut}(\overline{a^0a^1}/\overline{a^0, a^1})$ is the inverse limit of the groups $\text{Aut}(\text{Mor}_\mathcal{G}_i(a^0_i, a^1_i)/\overline{a^0, a^1})$ under the natural homomorphisms
\[
\rho_{j,i} : \text{Aut}(\text{Mor}_\mathcal{G}_j(a^0_j, a^1_j)/\overline{a^0, a^1}) \to \text{Aut}(\text{Mor}_\mathcal{G}_i(a^0_i, a^1_i)/\overline{a^0, a^1})
\] induced by the fact that $\text{Mor}_\mathcal{G}_i(a^0_i, a^1_i)$ is in the definable closure of $\text{Mor}_\mathcal{G}_j(a^0_j, a^1_j)$ when $j \geq i$.

By the way we defined our theory $T$, we can select a system of group isomorphisms $\lambda_i : H_i \to \text{Mor}_\mathcal{G}_i(a^1_i, a^1_i)$ for $i \in I$ such that the following diagram commutes:

\[
\begin{array}{ccc}
H_j & \xrightarrow{\varphi_{j,i}} & H_i \\
\downarrow \lambda_j & & \downarrow \lambda_i \\
\text{Mor}_\mathcal{G}_j(a_j, a_j) & \xrightarrow{\chi_{j,i}} & \text{Mor}_\mathcal{G}_i(a_i, a_i)
\end{array}
\]

To finish the proof of the Proposition, it is enough to find a system of group isomorphisms
\[
\sigma_i : H_i \to \text{Aut}(\text{Mor}_\mathcal{G}_i(a^0_i, a^1_i)/\overline{a^0, a^1})
\]
such that the following diagram commutes:

\[
\begin{array}{ccc}
H_j & \xrightarrow{\varphi_{j,i}} & H_i \\
\downarrow{\sigma_j} & & \downarrow{\sigma_i} \\
\text{Aut}(\text{Mor}_{G_i}(a_j^0, a_j^1)/\overline{a^0}, \overline{a^1}) & \xrightarrow{\rho_{j,i}} & \text{Aut}(\text{Mor}_{G_i}(a_i^0, a_i^1)/\overline{a^0}, \overline{a^1})
\end{array}
\]

(Then by the discussion above, \(\text{Aut}(\overline{a^0}a^1/\overline{a^0}, \overline{a^1})\) will be isomorphic to the inverse limit of the groups \(H_i\), which is \(G\).)

We define the maps \(\sigma_i\) so that for any \(h \in H_i\) and any \(g \in \text{Mor}_{G_i}(a_i^0, a_i^1)\),

\[\sigma_i(h) = \lambda_i(h) \circ g.\]

(Note that this rule determines a unique elementary permutation of \(\text{Mor}_{G_i}(a_i^0, a_i^1)\) fixing \(\text{acl}(a_i^0) \cup \text{acl}(a_i^1)\) pointwise.) This is a group homomorphism since

\[\sigma_i(h_1 h_2) = \lambda_i(h_1) \circ \lambda_i(h_2) \circ g = [\sigma_i(h_1) \circ \sigma_i(h_2)](g).\]

Clearly \(\sigma_i\) is injective, and it is surjective because of the following:

**Claim 3.8.** For any \(f\) and \(g\) in \(\text{Mor}_{G_i}(a_i^0, a_i^1)\), there is a unique elementary permutation \(\sigma\) of \(\text{Mor}_{G_i}(a_i^0, a_i^1)\) sending \(f\) to \(g\) and fixing \(\text{acl}(a_i^0) \cup \text{acl}(a_i^1)\) pointwise.

**Proof.** If \(f = h \circ g\) for \(h \in \text{Mor}_{G_i}(a_i^1, a_i^1)\), then \(\sigma(f')\) must equal \(h \circ f'\) for any \(f' \in \text{Mor}_{G_i}(a_i^0, a_i^1)\).

Finally, we must check that the maps \(\sigma_i\) commute with \(\varphi_{j,i}\) and \(\rho_{j,i}\). Pick any \(j \geq i, h \in H_j\) and \(f \in \text{Mor}_{G_i}(a_i^0, a_i^1)\). On the one hand,

\[\rho_{j,i}(\sigma_j(h))(f) = \chi_{j,i}(\sigma_j(h)(f')) = f\]

where \(\chi_{j,i}(f') = f\).

On the other hand,

\[\varphi_{j,i}(\lambda_j(h))(f) = \chi_{j,i}(\lambda_j(h)) \circ \chi_{j,i}(f') = \chi_{j,i}(\lambda_j(h)) \circ f.\]

These last two equations show that \(\rho_{j,i} \circ \sigma_j = \sigma_i \circ \varphi_{j,i}\), as desired.

**Remark 3.9.** These examples also show that homology groups of types are not always preserved by nonforking extensions. In the example above, if \(A\) is some algebraically closed parameter set containing a point in \(p(C)\) and \(q\) is the nonforking extension of \(p\) over \(A\), then \(q\) has 4-amalgamation, and so by Fact 1.8 above, \(H_2(q) = 0\).
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