5.2. Tessellating Regular Polygons

Regular Tessellations. In the preceding section we concluded that every triangle and every quadrilateral tessellate the plane. Thus, the two regular polygons, the equilateral triangle and the square tessellate the plane. Moreover, we noticed that the regular pentagon does not tessellate the plane. Now, what about the next regular polygon? Does the regular hexagon tessellate the plane? The interior angle measure of the regular hexagon is 120°. Why? 360° is divisible by 120°! Therefore we can arrange three congruent regular hexagons around a point with no gaps or overlaps (Figure 5.2.1). This means the regular hexagon tessellates the plane (Even bees knew this!).

We observe that as we continue and examine regular polygons with more sides than the regular hexagon, their interior angle measures increase and will be more than 120°. We also notice that in order to fill the space around a point we need at least three copies of a regular polygon. Thus, none of the regular polygons with more sides than the hexagon will tessellate the plane.

The tessellation, which is created by a regular polygon, is called a regular tessellation. There are only three regular tessellations. These tessellations are created by equilateral triangles, squares, and regular hexagons (see Figure 5.2.2).

There are interesting problems that can be posed using any of these regular tessellations. For instance, let us consider polygons that have been created from a set of four connected congruent squares (Figure 5.2.3). They are called tetrominoes. There are five tetrominoes. We can try them to see if each tetrominoe can tessellate the plane.
Figure 5.2.3

**Semiregular Tessellations.** After monohedral construction of tessellations using regular polygons we are now interested in studying the *polyhedral tessellations* of regular polygons: a tessellation constructed by combinations of two or more regular polygons.

To determine the possibility of such a tessellation, we should first study their arrangements around a vertex. For example two squares and three equilateral triangles can be positioned around a common vertex without any gaps or overlaps simply because the sum of their angle measures is 360°. Determining the angle measure of regular polygons enables us to determine which combinations will fill out the space around a point. We also notice that in general the following limitations exist for such a combination:

1. There cannot be less than three regular polygons around a vertex.
2. There cannot be more than six regular polygons around a vertex (The most is six equilateral triangles).
3. There cannot be more than three different types of regular polygons around a vertex (If we arrange an equilateral triangle and a square around a common vertex, the remaining degrees will be 210° which is not enough for two more different regular polygons).

Therefore, we can find all possible arrangements of regular polygons that will fill the space around a point without gaps or overlaps (Figure 5.2.4).
Each number in any of the sequences in Figure 5.2.4 represents a regular polygon. Each sequence represents an arrangement of polygons around the common vertex. For example, 3.6.3.6 means if we start from a triangle and proceed sequentially around the common vertex, we will meet a hexagon, a triangle, and finally a hexagon in a row. However, 3.3.6.6 is a different arrangement and indicates that there are two adjacent triangles and two adjacent hexagons.

To construct a tessellation of combinations of regular polygons it is necessary that the combination be one of the above 21 arrangements. But to your surprise, unlike the monohedral case, it is not true that each of the above combinations results in a tessellation! For example 3.4.4.6 cannot produce a tessellation (Do you have some tessellation foam blocks? Why not try it!).

We should mention that it is possible to create an infinite number of tessellations of combinations of regular polygons, each having different arrangements of polygons around their vertex points. However, for our course of study, we are only interested in those tessellations in which all the vertex points have identical sets of polygons in the same order.

It can be shown that there are only eight tessellations that are formed by combinations of two or more regular polygons such that the arrangement of polygons at every vertex point is identical (Figure 5.2.5). They are called semiregular tessellations.

Therefore, semiregular tessellations are tessellations that

1. **Are formed by combinations of two or more regular polygons.**

2. **Have identical arrangements of polygons at their vertex points.**

The three regular and eight semiregular tessellations are called *Archimedean Tilings.*
Figure 5.2.5. The eight semiregular tessellations
The Dual of a Tessellation. As we studied in Chapter Four every regular polygon has a center of rotation. Consider the regular triangular tessellation. Now, mark all these centers and connect them as is shown in Figure 5.2.6.a to create another tessellation. This new tessellation that is identified by dashed lines is called the dual of a regular triangular tessellation. We notice that it is a regular hexagonal tessellation. Therefore, the dual of a regular triangular tessellation is a regular hexagonal tessellation. It is not difficult then to conclude that in reverse, the dual of a regular hexagonal tessellation is a regular triangular tessellation. We also notice that the square tessellation is self-dual (Figure 5.2.6.b).

![Figure 5.2.6](image)

We may use \( \{p, q\} \) notation to represent a regular tessellation where \( p \) and \( q \) are integers. The number \( p \) indicates the number of sides of the polygon and \( q \) shows the number of copies of the polygon about each vertex point. For example, \( \{3, 6\} \) is the regular triangular tessellation. The dual of the tessellation \( \{3, 6\} \) is \( \{6, 3\} \), and the dual of the tessellation \( \{4, 4\} \) is \( \{4, 4\} \).

Exercise Set 5.2

1. Which regular polygons can generate monohedral tessellations?

2. What is the dual of a tessellation? Identify each regular tessellation using \( \{p, q\} \) notation and represent its dual.

3. What is a semiregular tessellation? How many semiregular tessellations exist?

4. What is the minimum number of regular polygons surrounding a vertex point? What is the maximum number of regular polygons surrounding a vertex point? Explain.

5. What are the Archimedean Tilings? Explain.

6. In Chapter Four we found that in a regular \( p \)-gon, if \( m \) is its interior angle measure, then \( m = (p - 2)180°/p \). From this formula, we can obtain \( p = 360°/(180° - m) \), \( m < 180° \), where \( p \) is a positive integer. Consider that we want to fill out the space around a point using exactly three regular polygons where one of them is an equilateral triangle. Find all possible cases.

7. Using formulas presented in Problem 6, find all possible cases for combinations of three different regular polygons, where one of them is a square, which can fill out the space around a point.
8. Using Sketchpad choose “Show Grid” in Graph menu and then hide the two axes that will appear. Now for each tetrominoe (Figure 5.2.3), using functions in the Transform menu, determine if it tessellates the plane.

9. Using Sketchpad add grid and perform the following:

   (a) Construct all twelve pentominoes (Five connected congruent squares).

   (b) Check to see if each pentomino tessellates the plane.

10. A pentiamond is a figure that is made up of five connected equilateral triangles. Using Sketchpad perform the following:

    (a) Construct each pentiamond (There are four of them). You may construct one equilateral triangle and then use appropriate functions from the Transform menu for this purpose.

    (b) Construct the regular triangular tessellation and display it using dashed line. Now try each pentiamond using a thick line to see if it tessellates the plane.

11. Construct the regular hexagonal tessellation in Sketchpad. Then construct its dual.

12. Construct the semiregular tessellation 3.3.3.3.6 in Sketchpad and then construct its dual.

13. Construct the semiregular tessellation 3.3.4.3.4 in Sketchpad and then construct its dual.

14. Construct the semiregular tessellation 4.8.8 in Sketchpad and then construct its dual.

15. Construct the semiregular tessellation 3.6.3.6 in Sketchpad and then construct its dual.

16. Consider the regular triangular tessellation. Choose one of its tiles. We observe that the following statements are true:

    (a) The center of the triangle is also a center of three-fold rotational symmetry for the tessellation.

    (b) Each vertex of this triangle is a center of six-fold rotational symmetry for the tessellation.

    (c) Each midpoint of a side of this triangle is a center of two-fold rotational symmetry for the tessellation.

Based on the above observation, explain similar cases for the square and hexagonal tessellations.