The Consumption-Savings Decision and Credit Markets

In this section we change our perspective to intertemporal decisions and tradeoffs that occur over time. The key component to intertemporal choice is understanding the consumer’s consumption-savings decision. This decision is dependant on the tradeoff that we face between current and future consumption. Another important concept is the Ricardian equivalence theorem which states that under certain conditions the size of the government’s deficit is irrelevant. To study these decisions we build a simple two period model of the economy. The key variable in this model will be the real interest rate at which consumers and the government borrow and lend.

1 A Two-Period Model of the Economy

In this section we set up a simple two period model of the economy with just consumers and a government later we will add in firms and investment.

1.1 Consumers

In this economy we are going to assume that consumers face two periods, today and tomorrow. Over their lifetime, the consumer receives income \( y \) today, and \( y' \) tomorrow.\(^1\) Throughout this section all primed variable denote tomorrow’s value. The consumer must make choices on the amount of consumption today, \( c \), and the amount of consumption tomorrow, \( c' \). The consumer also chooses savings today, \( s \), to carry forward to tomorrow that can be added to tomorrow’s consumption. We are going to assume that the gross return on savings is \((1 + r)\). Given their income stream, consumers must choose \( c \), \( s \) and \( c' \) over their lifetime.

In each period, the consumer faces a budget constraint. This budget constraint simply says that in each period, a consumer’s expenditures can not exceed existing wealth. Since this household lives for two periods, we can argue that our consumer faces two constraints. The constraints are written as

\[ c + s = y - t \]

\(^1\)We are abstracting away from any labor-leisure choices. The consumer receives this income exogenously. We will discussion some idea associated with labor-leisure choices when we cover business cycles.
for the first period and
\[ c' = (1 + r)s + y' - t' \]
for the second period.

The consumer’s consumption can not exceed their wealth in any period. In the first period, the consumer can only consume from the today’s income. In the second period they can consume tomorrow’s income and any savings that they brought from the first period. Note: savings can be positive or negative. If \( s < 0 \), this simply means that the consumer borrowed for higher first period consumption. In period two, this consumer will have to pay back the loan at rate \( (1 + r) \), and give up some consumption tomorrow. If \( s > 0 \), this simply means that the consumer saved some of his period one resources for higher period two consumption. In period two, this consumer will earn a gross return of \((1 + r)\) on their savings, and have higher consumption in period two.

1.1.1 The Consumer’s Lifetime Budget Constraint

We could work with the consumer’s consumption-saving decision in the current framework, however, we can simplify the problem by combining the two period specific budget constraints into a single budget constraint over both periods. Notice that \( s \) is in both budget constraints. If we solve period two’s constraint for \( s \), we find
\[ s = \frac{c' - y' + t'}{1 + r} \]
We can simply plug this expression into the first period budget constraint and get the consumer’s lifetime budget constraint,
\[ c + \frac{c' - y' + t'}{1 + r} = y \]
Thus the consumer’s problem is now simply a choice of consumption today and consumption tomorrow. It will be convenient to rearrange this lifetime constraint in the following way
\[ c + \frac{c'}{1 + r} = y + \frac{y'}{1 + r} - t - \frac{t'}{1 + r} \]
This states that the lifetime present value of consumption must equal the lifetime present value of income minus the present value of lifetime taxes.
This can also be called the lifetime present value of disposable income. Given this restriction, the consumer simply chooses \( c \) and \( c' \) to make himself as well off as possible.

We can label the present value of lifetime disposable income as **lifetime wealth**, \( we \), since this is the quantity of resources that the consumer has available to spend on consumption over his lifetime. So the lifetime budget constraint can be written as,

\[
we = c + \frac{c'}{1+r}
\]

If we rearrange this so that tomorrow's consumption is on the left hand side, we get

\[
c' = (1 + r)we - (1 + r)c
\]

This is a simple linear relationship which we can graph. This is a line with y-intercept \((1 + r)we\) and slope \(-(1 + r)\).

The region inside the budget constraint represents all feasible choices for the consumer. The slope of the budget set is determined by the real interest rate \(-(1 + r)\). There is a point \( E \) that represents the **endowment point**. This choice corresponds to the situation where the consumer simply eats his disposable income every period. If the consumer chooses any points above \( E \), then today consumption \( c \) is less than today’s disposable income \( y - t \). Thus this person has chosen to keep positive savings. This consumer is a lender to the saving market. On the other hand, points below \( E \) represent situations where the consumer chooses a level of consumption today that exceeds today’s income. Therefore, this consumer must borrow in order to obtain this higher current consumption.

### 1.1.2 The Consumer’s Preferences

What is the objective of the consumer? They want to maximize lifetime utility. When dealing with consumer preferences we will assume the following properties.

1. **More is always preferred to less.** The more consumption whether is occurs today or tomorrow always makes the consumer better off.

2. **Consumers prefer diversity in the consumption bundle.** Consumers have a preference towards smoothing consumption. Namely, a consumer would not like a situation where consumption is very high in one
period and very low in the other period. This does not mean that the consumer will choose the same consumption each period.

3. Current consumption and future consumption are normal goods. If there is a parallel shift in the consumer’s budget constraint, then current and future consumption will both increase. This is related to the desire to smooth consumption over time. If there is a shift in the budget constraint then lifetime wealth, we, has increased. The consumer will choose to spread in increased wealth across both time periods leading to higher current and future consumption.

The best way to graphically depicts a consumer’s preferences is by constructing an indifference map. The indifference map is a family of indifference curves. An indifference curve connects a set of points, these points represent consumption bundles among which the consumer is indifferent.

The consumer is willing to substitute. More specifically a consumer would be willing to substitute away some of today’s consumption for higher future consumption if it gets the consumer to a higher indifference curve.

**Definition 1** The Marginal Rate of Substitution (MRS) of today’s consumption for future consumption, is the rate at which the consumer is just willing to substitute today’s consumption for more consumption tomorrow.

Graphically the MRS for an allocation is the negative of the slope of the indifference curve at the chosen allocation. The MRS tells us how easy it is to get the consumer to substitute today’s consumption for higher consumption in the future.

**1.1.3 Consumer Optimization**

The consumer’s problem is to choose current and future consumption that achieves the highest indifference curve subject to the consumer’s lifetime budget constraint. To solve this problem the consumer will choose the indifference curve that is tangent to the lifetime budget constraint. The point of tangency represents the consumer’s optimal choice. At this point the slope of the indifference curve, at the optimal choice, is equal to the slope of the budget constraint. At the optimal choice the slope of the highest attained
indifference curve will equal the slope of the lifetime budget constraint which means

\[-MRS = -(1 + r)\]  
\[MRS = (1 + r)\]

The consumer’s marginal rate of substitution for future consumption is equal to the relative price of current consumption in terms of future consumption. The consumer optimizes by choosing the consumption bundle on his lifetime budget constraint where the rate at which he is willing to trade off current consumption for future consumption is the same as the rate at which he can trade current consumption for future consumption in the market by saving.

Let the point A represents the consumer’s optimal consumption bundle where he consumes \(C^*\) today and \(C'^*\) tomorrow. The slope of the indifference curve at A is equal to the slope of the budget constraint. If the endowment starts at a point above A, then this consumer prefers to be a borrower. He wishes to spend more than his current income and borrow and the expense of future income. If the endowment would have started below A, we would say that this consumer chooses to be a lender. He chooses not to consume all of his current income. Instead he decides to save part of it, and in return gets higher future consumption.

### 1.1.4 Gains from Saving

The main insight that is gained from the simple two-period Fisher model is that saving creates utility gains for the consumers. Suppose that this consumer is not given access to a means of savings. This means that the consumer would be forced to consume his endowment and remain at point E. At this point the consumer receives utility level that goes through E. Now consider the situation where the consumer is given access to savings markets. In this environment, the consumer is no longer forced to consume at the endowment level. He may choose to borrow or save with the goal of reaching a higher utility level. Given perfectly functioning markets, this consumer will choose to consume \(C^*\) today and \(C'^*\) tomorrow. At this allocation, the consumer can achieve a higher indifference curve that is tangent to the budget constraint, \(I_2\). Thus, the ability to save generates a utility gain for the consumer.
1.1.5 An Increase in Current-Period Income

Suppose, holding all else equal, the consumer receives higher income in period 1. How would this change the consumer’s decisions on $c$, $c'$, and $s$? Let’s suppose period 1 income increases from $y_1$ to $y_2$. The result of this change is an increase in lifetime wealth from

$$we_1 = y_1 + \frac{y'}{1 + r} - t - \frac{t'}{1 + r}$$

to

$$we_2 = y_2 + \frac{y'}{1 + r} - t - \frac{t'}{1 + r}$$

This is a simple change in the intercept of the budget constraint which shifts to the right by the distance $y_2 - y_1$. Because current and future consumption are normal goods, the new optimal consumption bundle will be located up and to the right of the original bundle. What about the savings decision. We can look at this from the first period budget constraint.

$$c + s = y - t$$

Given that $t$ did not change and the change in $y$ is bigger than the change in $c$ we can see that the change in $s$ must also be positive. Thus an increase in current income increases current consumption, future consumption, and savings. This occurs because of the desire for consumption smoothing.

1.1.6 An Increase in Future Income

So what happens if there is an increase in future income. Basically it the the same kind of effect, the endowment point moves up vertically and the budget constraint moves out by the distance of the income change. Consumption will increase in both periods. The main difference is that savings will decrease instead of increase, the increase in future income decreases the need to save for the future.

1.1.7 Temporary and Permanent Changes in Income

This theory is called the permanent income hypothesis. In this section, we are going to discuss the theory behind the permanent income model and its implications.
Defining the Permanent Income Hypothesis  The permanent income hypothesis is built out of the following definitions and assumptions

1. An individual’s income $y_t$ in any period can be expressed as the sum of two components,

$$y_t = y^*_t + y^T_t$$

where $y^*_t$ is his permanent income and $y^T_t$ is his transitory income.

2. Actual consumption during the period can also be broken into a permanent and transitory part

$$c_t = c^*_t + c^T_t$$

3. Permanent consumption is proportional to permanent income. Let, $\lambda_i$ represent the factor of proportionality of permanent consumption with permanent income. This factor depends on the age, family structure, and other life-cycle influences that affect the consumption behavior of the consumer. Permanent consumption depends on and only on permanent income. It is not related to transitory income. Thus for permanent consumption the consumption function is,

$$c^*_t = \lambda_i y^*_t$$

4. Transitory consumption is not systematically related to either permanent or transitory income. It is a random disturbance.

Given these assumptions and definition we can state the total consumption function as

$$c_t = \lambda_i y^*_t + c^T_t$$

If we wish to look into the consumer’s propensity to consume, we find that this consumption function implies that the marginal propensity to consume out of the permanent income is $\lambda_i$, and the propensity to consume out of the transitory income is zero. This key characteristic helps to explain an empirical trait, namely there is a distinct difference in the marginal propensity to consumption in the short run versus the long run. Specifically, short run propensities to consume are generally much lower than ones in the long run.
Long run consumption has slope $\lambda_i$ and can be thought of as the permanent consumption function. Suppose we want to survey the consumers about their consumption. We are going to take two looks at their behavior one now in period 1, and one later in period 2. In period 1, the permanent income of a typical household is $y^*(1)$, and the typical permanent consumption would then be $c^*(1)$. However given that there is a transitory component in total consumption. The actual consumption that we may measure could be above or below the permanent consumption level.

Also, we must remember that income contains a transitory part. It is entirely possible that we could find a luck person who earns $y_H$ instead of $y^*(1)$. Therefore, we could find consumption equal to permanent consumption, but because of the transitory component of income we could be at points above or below permanent income.

If we simply look at our current situation, we would find that the best fitting consumption-income combinations would occur along a flat line. This would most likely generate the best consumption function for the group. This consumption function would have an intercept near $c^*(1)$ and have a marginal propensity to consume near zero.

If we decide to take a second survey in the future, we would find that for the second point in time would be another flat line higher up the $y$ axis. Once again the short run propensity to consume is near zero. However, if we compare the average income and average consumption over the two surveys we find that the best fitting consumption function over the long run, and the propensity to consume would then be $\lambda_i$. There would be very little difference in the average rate of consumption to income over the two surveys.

**Permanent Income Hypothesis in Context of the Two Period Model**

We can simulate Friedman’s idea of Permanent (Planning) Income in the two period model. To mimic a positive shock to temporary income we can simply increase current income and we get the same results as earlier. The mimic a positive shock to permanent income we could increase income in both the current and future income. This generates a much larger shift in the budget constraint and thus permanent income effects are larger than transitory. Start thinking ahead about government finance. Suppose the government proposes a permanent versus temporary tax cut, how would consumers react?
1.1.8 An Increase in the Real Interest Rate

What would happen to this situation is the rental rate, \( r \), were to increase? Remember we can write the lifetime budget constraint as,

\[ c' = (1 + r)we - (1 + r)c \]

We see that with an increase in \( r \), we raise the intercept and raise the slope of the budget set. An increase in the rental rate of capital will cause the budget constraint to become steeper. This graph assumes that the household only receives wages in the current period and no wages in the future period. The endowment point is on the horizontal axis. In general, changes to the interest rate lead to rotations in the budget constraint around the endowment point. Why?? This can lead to substantial changes in the optimal choices for the consumer. Regardless of the situation, an increase in the rental rate of capital makes a higher indifference curve attainable. There are other situations that can cause the budget set to move, we will leave these homework problems.

1.2 Government

Now that we have looked at the consumers let’s complete the description of this economy by looking at the government. In this economy the will purchase good \( G \) in the current period and \( G' \) in the future period. These purchases are partial funded by taxes today \( T \) and taxes tomorrow \( T' \). Given there are \( N \) consumers we know that \( T = Nt \) and \( T' = nt' \). The government, just like the households, can borrow in the current period by issuing bonds. They borrow at the real interest rate \( r \). If we let \( B \) be the quantity of government bonds. The current period government budget constraint is

\[ G = T + B \]

the future period constraint is

\[ G' + (1 + r)B = T' \]

Just like the households these period by period constraints can be formulated into a lifetime budget constraint by substituting \( B \) out of the future constraint into the current constraint. We find

\[ B = (T' - G')/(1 + r) \]
and then by substituting we find

\[ G + \frac{G'}{1+r} = T + \frac{T'}{1+r} \]

One important point here is that the government must eventually pay off their debts.

### 1.3 Competitive Equilibrium

We have now described the economy and can now describe the solution for the model by explaining the competitive equilibrium. The key market in this economy is the credit market which is where the government and the households interact. Both the consumers and the government can borrow and lend at the market interest rate. The competitive equilibrium in this two-period economy consists of three conditions.

1. Each consumer choose current and future consumption and savings optimally given market interest rate \( r \)
2. The government lifetime budget constrain holds.
3. The credit market is in equilibrium

The equilibrium is in equilibrium when the quantity that the consumers want to lend in the current period is equal to what the government wants to borrow. That is equilibrium in the credit market is

\[ S = B \]

This implies

\[ Y = C + G \]

Why? Well think about national income account. From the household budget constraint we have

\[ S = Y - C - T \]

from the government constraint we have

\[ B = G - T \]
Since equilibrium implies \( B = S \),
We can substitute to get
\[
Y - C - T = G - T
\]
or
\[
Y = C + G
\]

2 Ricardian Equivalence Theorem

Earlier we stated that an increase in government spending come at a cost of crowding out private consumption. However, because \( G = T \) in that example, we could not figure out where the costs were coming from the spending or the taxes. With government borrowing introduced into this economy, we can now look at spending and tax changes separately. This allows us to look at the Ricardian Equivalence Theorem. This theorem states that a change in the timing of taxes by the government is neutral. Neutral means that in equilibrium a change in current taxes is exactly offset by change in future taxes and has no effect on the real interest rate or optimal consumption behavior. More or less government deficits do not matter. What we will find is that deficits do matter, this Theorem is just a good starting point into seeing why. Why does the Ricardian Equivalence Theorem hold in this economy? First since every consumer shares and equal load of the tax burden, we can easily rewrite the government budget constraint as
\[
G + \frac{G'}{1 + r} = Nt + \frac{Nt'}{1 + r}
\]
which can be rewritten as
\[
t + \frac{t'}{1 + r} = (1/N)[G + \frac{G'}{1 + r}]
\]
which says that the present value of taxes for a consumer are equal to it’s share of the present value of government spending.

We can thus substitute this into the consumer’s budget constraint giving us
\[
c + \frac{c'}{1 + r} = y + \frac{y'}{1 + r} - (1/N)[G + \frac{G'}{1 + r}]
\]
Looking at the right hand side of the budget constraint we can see that if taxes are altered in a way that does not influence G or G’, consumption is unaffected since everything on the right hand side is unchanged by a change in the timing of taxes. Thus the theorem holds.

The timing of taxes has no influence on consumption, but it does alter one decision: savings. A decrease in current taxes increases current disposable income, but does not influence consumption. All of the increase in income is chosen to be passed on as increased savings which, given the credit market clear, is exactly offset by an increase in government borrowing.

2.1 Ricardian Equivalence: A Numerical Example

Consider an economy with 500 consumers who are all identical with an equilibrium interest rate of 5 percent. Each consumer receives $y = 10$, and $y’ = 12$, in addition they pay taxes $t = 3$ and $t’ = 4$. Lifetime wealth for the consumers is

$$w = 10 - 3 + (12 - 4)/1.05 = 14.61$$

Suppose we know that optimal consumption bundle is $c = 6$ and $c’ = 9.04$. We can verify that this consumption bundle satisfies the lifetime budget constraint. That is,

$$6 - 9.04/1.05 = 14.61$$

Thus each consumer chooses $s = 10 - 6 - 3 = 1$ giving aggregate savings to be 500. The government purchases 2000 units in the current period and 1475 units in the future period. Because current taxes $T = 3\times500 = 1500$ and $T’ = 4\times500 = 2000$ in the future period, the government borrows $B = 500$ in the current period. Thus we have national savings equal to $500 - 500 = 0$. The government budget constraint is

$$2000 + 1475/1.05 = 1500 + 2000/1.05$$

We also know $Y = 10\times500 = 5000$ and aggregate consumption is $C = 6\times500 = 3000$. Thus we have $Y=C+G$ and the credit market clears. Now suppose the government reduces taxes in the current period to 2 units and increases taxes in the future to 5.05 units. Suppose also the interest rate is unchanged at 5 percent. Then the government constrain still holds as,

$$2000 + 1475/1.05 = 1000 + 2525/1.05$$
In addition consumer wealth is unchanged at

\[ w = 10 - 2 + (12 - 5.05)/1.05 = 14.61 \]

given this the consumer will still want to consume \( c = 6 \) and \( c' = 9.04 \). Aggregate private savings has now increased by the size of the tax cut to 1000 and government savings has decreased by 500 to -1000 which still generates the equilibrium national savings at 0.

### 2.2 Ricardian Equivalence: A Graph

Graphically the Ricardian Equivalence Theorem is simply shown as a movement in the endowment point. Suppose the government decreases current taxes thus implying and increase in future taxes. This decrease in current taxes increases current disposable income and moves the endowment point down the budget constraint. Since there is no shift in the constraint, there is no change in the optimal allocation \((c, c')\). Thus the timing of taxes has no implication on utility even though there is a tax cut in the current period. Tax cuts are not a ‘free lunch’ decreases in current taxes imply larger deficits which must be paid with larger tax burdens in the future period.

### 2.3 Ricardian Equivalence and Credit Market Equilibrium

In the current environment a change in taxes in the current period lead to an equivalent move in private savings and a equal but offsetting moving in government borrowing. Thus, the credit market remains in equilibrium at the prevailing interest rate. Thus gives us two implications about tax cuts. First, a decrease in current taxes need not generate a large increase in current consumption which lines up with the permanent income idea from Freidman. Second, tax cuts are not a free lunch, tax cuts in the current period imply future tax increases to pay off the increase in debt generated from the original tax cut.

### 2.4 Ricardian Equivalence and the Burden of Government Debt

For individuals debt represents a liability that reduces lifetime wealth. The Ricardian equivalence theorem implies that the same logic holds for gov-
ernment debt which measures our future tax liabilities. Now in the current environment the burden of debt is spread equally across all consumers. In reality, may fiscal issues can complicate this assumption. To understand this let’s look at four key assumptions we made in this environment.

1. In our example we assumes that tax changes were the same for everyone in the present and the future. If some consumers received higher tax cuts than others, then lifetime wealth could change for some consumers. This would generate changes in consumption choices and could change the equilibrium interest rate. This holds true for both current and future taxes which could be shared unequally across the population.

2. Another assumption is that any government debt that is generated during the consumer’s lifetime will be paid off in that lifetime. In reality the government can differ future taxes increased to pay off the debt for a LONG time. So it is conceivable that some consumers could receive the benefit of a tax cut in the current period and once the higher future taxes are induced the consumer is either retired or dead. This possibility increases with the age of the consumer. This shifts the burden of debt onto the young and off the old creating a generational redistribution of wealth.

3. We also assumed that taxes are lump sum, which is not really used in practice. All other forms of taxes tend to generate distortions change the relative prices of goods and welfare loss.

4. The fourth assume is that there are perfect credit markets in that consumers can borrow and lend as much as the want given their budget constraint. In addition we assume borrowing and lending are done at the same rate. In reality, consumers do face constraints on the amount they can borrow and typically the borrowing interest rate is higher than the lending interest rate. Also the government tends to borrow at rates below that of consumers. These credit market imperfections, can alter some consumer behavior. Some credit constrained individuals could be affected beneficially by a tax cut as these consumers are at their endowment point and a tax cut shifts their endowment point down and to the right pushing them to a higher indifference curve that is still below the utility level of a non constrained consumer. (DRAW IT)
The main theme to pull out of the Ricardian theorem is that changes in current taxes have consequences for future taxes.

3 Moving Beyond Two Periods: The Life-Cycle Model

Most interesting life-cycle saving questions cannot be addressed in a two period framework. Thus we need to extend this basic model to allow for multiple periods. Well, we could continue our current approach, but it gets very difficult to draw things in more than two dimensions. In some life-cycle problems we may want to extend the model to cover 50 or 60 periods. Drawing a 50 dimensional object is a little tricky. Our only other solution is to derive an algebraic equivalent to the graphical model. Specifically we need to derive expressions for the lifetime budget constraint and the indifference map.

3.1 The Preference Function

We can represent the preferences of a consumer by using an utility function. This function will rank all possible consumption bundles and give each bundle a number corresponding to what we call units of happiness. The consumer will then simply be trying to choose the consumption bundle that generates the most units of happiness, they simply will want to maximize the utility function. These multi-period utility functions take on the typical form of

$$\log(c_1) + \beta \log(c_2) + \beta^2 \log(c_3) + \beta^3 \log(c_4) + \ldots$$

where $\beta$ is called the subjective discount factor. This factor measures how much less consumers value future utility. The range for $\beta$ is typically between zero and one. If $\beta$ is less than one, then the consumer values consumption today more than consumption tomorrow. A $\beta$ greater than one has the opposite effect, the consumer would place greater value on future consumption. Consumers with a $\beta$ less than one are said to have a positive rate of time preference.
3.2 The Opportunity Set

We can construct the lifetime budget constraint in a manner similar to the way we constructed it in the two period model. The easiest way to construct the new lifetime budget constraint is to start in the last period and work backwards. Since period can not borrow in the last period and there are no bequest motives in this model, the last period budget constraint is fairly straightforward. The consumption should equal the last period income plus any savings brought into the period. Then we simply solve the constraint for the savings variable and then move forward one period. Now this period becomes the last period and we can repeat this process until we reach period 1.

We are going to work through the example for a 3 period model. In period 3, the period budget constraint must be

\[ c_3 = (1 + r)s_2 + y_3 \]

where \( y_3 \) is the current period income, \( s_2 \) is the savings brought through from the last period, and \( r \) is the return on savings. We can solve this constraint for \( s_2 \) to get

\[ s_2 = \frac{c_3}{(1 + r)} - \frac{y_3}{(1 + r)} \]

We can then plug this into the period 2 budget constraint to get

\[ c_2 + \frac{c_3}{(1 + r)} - \frac{y_3}{(1 + r)} = (1 + r)s_1 + y_2 \]

Which we can rearrange as

\[ c_2 + \frac{c_3}{(1 + r)} = (1 + r)s_1 + y_2 + \frac{y_3}{(1 + r)} \]

We can now solve this equation for \( s_1 \)

\[ s_1 = \frac{c_2}{(1 + r)} - \frac{c_3}{(1 + r)^2} - \frac{y_2}{(1 + r)} - \frac{y_3}{(1 + r)^2} \]

and plug this expression into the 1st period budget constraint to getting,

\[ c_1 = -\frac{c_2}{(1 + r)} - \frac{c_3}{(1 + r)^2} + \frac{y_2}{(1 + r)} + \frac{y_3}{(1 + r)^2} + y_1 \]

\[ c_1 + \frac{c_2}{(1 + r)} + \frac{c_3}{(1 + r)^2} = y_1 + \frac{y_2}{(1 + r)} + \frac{y_3}{(1 + r)^2} \]

This expression is the 3 dimensional counterpart to the two dimension constraint we found earlier, this can be generalized to \( n \) periods.
3.3 The steady state Consumption Function

Thus we can now interpret this algebraically as one where the consumer seeks to maximize his lifetime utility function

\[ \log(c_1) + \beta \log(c_2) + \beta^2 \log(c_3) + \beta^3 \log(c_4) \ldots \]

subject to his lifetime budget constraint

\[ c_1 + \frac{c_2}{(1 + r)} + \frac{c_3}{(1 + r)^2} + \ldots = y_1 + \frac{y_2}{(1 + r)} + \frac{y_3}{(1 + r)^2} + \ldots \]

The mathematics for solving this problem are beyond the scope of this class, but you should at least be comfortable and understand how the problem is setup. It can be shown that the results for the model using this utility function take the following form

\[
\begin{align*}
c_{i+1} &= c_i(1 + r) \\
c_{i+2} &= c_{i+1}(1 + r) = c_i(1 + r)^2 \\
c_{i+3} &= c_{i+2}(1 + r) = c_{i+1}(1 + r)^2 = c_i(1 + r)^3 \\
&\vdots \\
c_N &= c_i(1 + r)^{N-1}
\end{align*}
\]

where \( i \) represents the current period, and \( N \) represents the final period. We can them simply substitute these period consumption functions into our lifetime budget constraint to get the following result. Let \( W_i \) represent the total present value of wealth in period \( i \). Thus

\[ W_i = y_i + \frac{y_{i+1}}{(1 + r)} + \frac{y_{i+2}}{(1 + r)^2} + \ldots = c_i + \frac{c_{i+1}}{(1 + r)} + \frac{c_{i+2}}{(1 + r)^2} + \ldots \]

So if we substitute our consumption function into the right hand side of the preceding equation we find that

\[ W_i = c_i + c_i + \ldots + c_i = (N - i + 1)c_i \]

We can solve for period \( i \)'s consumption to get the typical result

\[ c_i = \frac{1}{(N - i + 1)} \times W_i \]
Then budgeted consumption in any period in this environment is proportional to total wealth at the start of the period, the factor of proportionally is equal to the total number of remaining periods. The life-cycle result simply states that consumers will consume a constant proportion of their wealth every period, agents seek to smooth consumption.

If you need to see the mathematics behind this result, you are directed to the Appendix in chapter 4. Numerical examples using this derived function are preformed in chapter 5.

3.4 Properties of the Life Cycle Model of Consumption

In this section we are going to probe further into the life cycle consumption function that we derived earlier. We are going to look at a numerical example using this consumption function, look into some properties of this function, and see how shocks can effect the consumption function.

3.4.1 Numerical Example

Let’s turn to a person a use a numerical example to derive their life cycle consumption and savings paths. For simplicity we will divide this person’s life span into 4, 20-year segments. The first segment represents childhood. In this period the individual receives no wage income, and his parents actually keep track of all his expenses and bill the individual at the end of the period. The next two segments will represent the working years, over this time the person will receive wage income. The last segment will correspond to retirement. The individual no long receives wage income, and must consume by sells assets that he has accumulated over the previous three periods.

As for labor earnings, suppose that the individual expects to receive wages of $300,000 over the 2nd period of his life, and $630,000 over the 3rd period of his life. Also, let’s assume that the current interest rate is %50 over the 20 year time horizon. This corresponds to roughly a 2% annually compounded interest rate.

The following table presents the life cycle solution for the consumption, saving, wealth, and components over the four time periods.
At the start of Period 1, you are born with no assets, so today’s assets are zero and you earn zero interest off of these assets. However, this person expects to make wages in the next two periods. At a 50% interest rate the present value of this future income stream is

\[ PV\ wages_1 = 0 + 300,000 \times \frac{1}{1 + 0.5} + 630,000 \times \frac{1}{(1 + 0.5)^2} = 480,000 \]

This makes up the person’s entire wealth in period 1. So, using the Life Cycle consumption function we can calculate this person’s consumption in period 1 as

\[ c_1 = \frac{1}{n_3 - i + 1} \times wealth_1 = \frac{1}{4 - 1 + 1} \times 480,000 = 120,000 \]

Since this person earns no wages or interest in period one, net income equals zero and thus savings is calculated as

\[ saving_i = income_i - consumption_i = 0 - 120,000 = -120,000 \]

This means that in period 2, this person will start with -120,000 in assets. In period 2, the person must pay interest on this 120,000 loan, thus interest income equals -60,000 with a 50% interest rate. Thus period 2 financial wealth is equal to

\[ -120,000 + (-60,000) = -180,000 \]
The human component of total wealth is the sum of the current wages of 300,000 and the present value of the wages he expects to receive in period 3, \[ PV \text{ wages}_2 = 300,000 + 630,000 \times \frac{1}{1 + .5} = 720,000 \]

Total wealth is the sum of financial and human wealth. Thus total wealth is,

\[ W_2 = -180,000 + 720,000 = 540,000 \]

Reapplying the consumption function we find period 2 consumption to be,

\[ c_2 = \frac{1}{n_3 - i + 1} \times \text{wealth}_2 = \frac{1}{4 - 2 + 1} \times 540,000 = 180,000 \]

Period 2 net income is the sum of period 2 interest income and wage income,

\[ \text{income}_2 = -60,000 + 300,000 = 240,000 \]

Therefore savings in period 2 are,

\[ \text{saving}_2 = 240,000 - 180,000 = 60,000 \]

The change in the asset position is thus a positive 60,000 leaving a debt of 60,000 at the end of period 2 going into period 3. This same pattern continues through period 4. Note that the life-cycle consumption function will always result in exactly zero assets at the end of the person lifetime. They will always consume all of their wealth in the last period of life.

3.5 Properties of the Life Cycle Consumption Function

The most obvious difference between the life-cycle consumption function and a Keynesian consumption function is that the life cycle consumption function is a function of age and wealth. The Keynesian counterpart only uses current income. Would would this say about consumption smoothing in an environment where income fluctuates? The life cycle consumption function will produces smoother consumption than a Keynesian function.

Another key difference between the life-cycle and Keynesian approach this that the Keynesian consumption function assumes that people will become
more thrifty as they get richer. The life-cycle function assumes consumption is strictly proportional to wealth, so that someone with twice the wealth has twice the standard of living. More specifically, the Keynesian function has a falling average propensity to consume, the life cycle has a constant average propensity to consume equal to

\[ \frac{1}{n^3 - i - 1} \]

3.6 How shocks affect the Life Cycle Consumption Function

When using any model, an important question is to see how the model reacts to changes in the underlying environment. Given these reactions we can prefer simple economic analysis and answer interesting questions. You have already done this in the simple supply-demand framework. The main point that should have been captured from that environment was that it was not important what the initial steady state price and quantity were. The interesting aspect was how this steady state changed when something happened in the economy. We want to perform a similar exercise with the life cycle model.

3.6.1 Response to Unanticipated Changes in Income and Wealth

Suppose, a person receives an unexpected change in their income and wealth during their lifetime, how would this shock change the results of the life cycle model? Consider the 2 period wage earning example from above. Suppose that we give this person 10 percent less wage income each period. Thus, period 2 wage income equals $270,000 and period 3 wage income equals $567,000. If this change is perfectly anticipated, then, from the consumption function, we would expect consumption to be 10 percent lower each period.

Suppose, however, that the lower wage is not anticipated at the start of period 1, but that the change had occurred at the start of period 2 after all period 1 decisions have been made. Thus, all of the period 1 decision would be the same as before, but period 2 decision would be altered. In the case of period 2 consumption, the financial wealth in period 2 remains $180,000, but the human component of wealth will be altered. Wages are
no only $270,000 and the present value of current and future wages becomes

\[ PV_{wages} = 270,000 + \frac{1}{1 + .5} \times 567,000 = 648,000 \]

Thus period 2 wealth is

\[ wealth_2 = -180,000 + 648,000 = 468,000 \]

And then consumption in period 2 is calculated as

\[ c_2 = \frac{1}{n_3 - i + 1} \times wealth_2 = \frac{1}{4 - 2 + 1} \times 468,000 = 156,000 \]

which is even lower than the 10 percent reduction caused by an anticipated decline in income $162,000. The overoptimism in the first period leads to a larger reduction in the following periods. The following table shows the consumption paths over the two possible scenarios.

<table>
<thead>
<tr>
<th>Period</th>
<th>C_t anticipated</th>
<th>C_t unanticipated</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>108,000</td>
<td>120,000</td>
</tr>
<tr>
<td>2</td>
<td>162,000</td>
<td>156,000</td>
</tr>
<tr>
<td>3</td>
<td>243,000</td>
<td>234,000</td>
</tr>
<tr>
<td>4</td>
<td>364,500</td>
<td>351,000</td>
</tr>
</tbody>
</table>

3.6.2 Response to a change in Financial Assets

What would happen to this individual if there was an unexpected drop in the stock market? Well, if someone is holding a substantial portion of their wealth in the market would find it no longer feasible to maintain the current level of consumption. Why? Well, this financial wealth is just another component of total wealth, the effect consumption will be directly proportional to the change in wealth. More specifically, he will lower current consumption by \( \frac{1}{n_3 - i + 1} \) times to fall in wealth, and be permanently on a lower consumption path.

We can basically think of changes in wealth as shifts in the consumption function. Remember, the consumption function takes the following form,

\[ c_i = \frac{1}{(N - i + 1)} \times W_i \]
We can then break wealth into its two components, human $h_i$ and assets $a_i$. To get

$$c_i = \frac{1}{(N - i + 1)} \cdot h_i + \frac{1}{(N - i + 1)} \cdot a_i$$

which is a linear function with an intercepts of $\frac{1}{(N - i + 1)} \cdot a_i$ and a slope of $\frac{1}{(N - i + 1)}$.

Thus anything that leads to a drop in $a_i$ can simply be interpreted as a downward shift in the consumption function. The intercept is lower, but the slope is unchanged. Note that this means the average propensity to consume out of labor income is unaffected by this type of change. For every dollar increase in income $h_t$ I expect to, on average, consume $\frac{1}{(N - i + 1)}$ of it.