Final Exam
Math 263
December 19, 2001

Name__________________________

Do all of your work on the blank paper provided. At the end of the exam, hand in your answers with this cover sheet. Include your name on all pages of your exam.

§1 Calculation

1. Write the contrapositive, converse, and inverse of "If a real number is greater than 2, then its square is greater than 4."

2. Rewrite the following statement formally using universal and existential quantifiers: "Somebody trusts everybody." Write the negation of the statement.

3. Evaluate $\sum_{m=0}^{127} \frac{3}{2^m}$.

4. Let $b_k = 3b_{k-1} + 1$ for all $k \geq 1$, and suppose that $b_0 = 1$. Find a formula for all b_k.

5. Let $X = \{1,2,4,5,6,7,9,10\}$. Define a binary relation $\#$ on $\mathcal{P}(X)$ as follows: For all $A, B \in \mathcal{P}(X)$, we say $A \# B$ if and only if the number of elements of A is the same as the number of elements of B. Determine if $\#$ is reflexive, symmetric, or transitive. Determine if $\#$ is an equivalence relation.

§2 Comprehension

6. The statement "If Howard can swim across the lake, then Howard can swim to the island." is true. Can Howard swim across the lake? Explain.

7. State precisely the principle of mathematical induction. State precisely the principle of strong mathematical induction. Label each.

8. What is the inverse of a function? Prove that every bijective function has an inverse function.

9. What is a binary relation? Find four binary relations from $\{a, b\}$ to $\{1,2\}$ that are not functions.

§3 Application

10. Prove or disprove: For all integers n, the number $n^2 - n + 11$ is prime.

11. Prove or disprove: The sum of any two rational numbers is rational.

12. Prove or disprove: For any integer $n \geq 2$, the number $n^2 - 3$ is not divisible by 4.

13. Prove or disprove: $\sqrt{2}$ is irrational.

14. Prove or disprove: $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}$ for all integers $n \geq 2$.

15. Prove or disprove: For all integers $n \geq 3$, we have $2n+1<2^n$.

16. Prove or disprove: For any sets A and B, we have $(A \cap B)^C = A^C \cup B^C$.

17. Let $A=\{(n,2n) : n \text{ is a positive integer}\}$, and let $B=\{(x,y) : x, y \text{ are integers, with } xy \text{ even}\}$. Prove or disprove: $A \subseteq B$.

18. Let $\Sigma=\{0,1\}$ and define $\ell : \Sigma^* \to \mathbf{Z}^{\text{nonneg}}$ by $\ell(s)=$ the length of s. Is ℓ one-to-one? Prove or give a counterexample. Is ℓ onto? Prove or give a counterexample.