Math 273
Final Examination
December 13, 2005

Name_____

All questions are worth an equal number of points. All work is to be done on the blank paper provided. At the end of the exam, please hand in this sheet, together with all of your work.

§1 Calculation

1. Evaluate:
   a. $\lim_{h\to 0} \dfrac{\sqrt{h^2+9}-3}{h^2}$.
   b. $\lim_{x\to 0} \dfrac{\cos x - 1}{x^2}$.
   c. $\lim_{x\to 0} \dfrac{e^x - 1}{\sin x - 1}$.
   d. $\lim_{x\to 0^+} x^{\sin x}$.
2. Differentiate
   a. $g(s) = \sqrt{s} - e^{2s} + \tan(s) + \ln(s)$
   b. $q(y) = \dfrac{(y^3 - 3y^2 + y)}{y^2 + 1}$
   c. $y(x) = \sqrt[4]{\dfrac{2 + x^2\cos(x)}{x^4 + x^2 + 5}}$
   d. $f(x) = x^{\sin x}$.
3. The graph of the curve $2(x^2+y^2)^2 = 25(x^2-y^2)$ is called a lemniscate. Find dy/dx. Find the equation of the tangent line to the curve that passes through $(1, 3)$.
4. Find the exact maximum and minimum values of the given functions on the indicated intervals
   a. $f(t) = 7t(\ln(t) - 9)$ on $[1, 8]$.
   b. $x(y) = 2y - \dfrac{15}{\sqrt[3]{y}}$ on $[1, 6]$
5. Consider the function $f(x) = x\ln x - 2x$.
   a. Find the domain of f.
   b. Find the asymptotes of f.
   c. Find the intervals on which f is increasing or descreasing.
   d. Find the local maxima and minima of f.
   e. Find the intervals on which f is concave up or concave down.
   f. Find the inflection points of f.
   g. Sketch the curve.

6. Evaluate:

   a. $\int \left(x^3 - \frac{1}{x} + \sin x\right)\, dx.$

   b. $\int xe^{x^2}\, dx.$

   c. $\int_{1/2}^{\sqrt{3}/2} \frac{dx}{\sqrt{1-x^2}}.$

   d. $\int_0^1 \frac{y}{1+y^4}\, dy.$

§2 Comprehension

7. What is the precise definition of limit? What is the intuitive definition of limit. Rigorously prove that $\lim_{x\to 2}(7x-6) = 8$.

8. What does it mean for a function to be continuous at a point? Prove that if $f(x)$ is differentiable at $x = a$, then f is continuous at $x = a$.

9. What is the precise definition of the derivative? Use the definition (only) to determine the derivative of $f(x) = 1/x$ at $x = a$.

10. What is the Mean Value Theorem? Apply the Mean Value Theorem to $f(x) = \sqrt{x}$ to prove that $\frac{4}{3} + \frac{x}{6} \le \sqrt{x} \le 1 + \frac{x}{4}$ for all $4 \le x \le 9$. Conclude that $7/3 \le \sqrt{6} \le 5/2$.

11. What is the definite integral? Use the definition (only) to evaluate $\int_0^4 x^2\, dx$.

12. What are the two forms of the Fundamental Theorem of Calculus? Prove one.

§3 Application

13. In the adiabatic compression of a gas, the pressure P and the volume V satisfy the relation $PV^\gamma = C$, where γ and C are constants.

    a. What is the rate of change of volume with respect to pressure?

    b. Consider an ideal gas ($\gamma = 1.4$) at a pressure of 1.8 atm with a volume of 0.6 L. What is the numerical value of the rate of change of volume with respect to pressure?

14. A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a speed of 2 ft/s, how fast is the angle between the top of the ladder and the wall changing when the angle is $\pi/4$?

15. A right circular cylinder is inscribed in a cone of height h and base radius r. Find the largest possible volume of the cylinder.

16. Since raindrops grow as they fall, their surface area increases and therefore the resistance to their falling increases. A raindrop has an initial downward velocity of 10 m/s and its downward acceleration is

$$a = \begin{cases} 9 - 0.9t & \text{if } 0 \le t \le 10 \\ 0 & \text{if } t > 10. \end{cases}$$

    If the raindrop is initially 500m above the ground, how long does it take to fall?