Final Examination
Math 275
May 18, 2000

Name________________________

Do all of your work on the blank paper that has been provided.

1. Let P be the point $(1,2,3)$, let Q be the point $(-1,-1,2)$, and let R be the point $(3,0,-1)$. What is the equation of the line through P and Q? What is the equation of the plane through P, Q, and R?

2. Let $\vec{A} = \langle 1,2,-2 \rangle$ and let $\vec{B} = \langle 3,-3,1 \rangle$. Find the angle between $\vec{A}$ and $\vec{B}$. Find a vector perpendicular to $\vec{A}$ and $\vec{B}$. Find the area of a parallelogram with one vertex at the origin, and sides along $\vec{A}$ and $\vec{B}$.

3. Graph the prolate cycloid $x(\theta) = \theta - 2\sin\theta$, $y(\theta) = 1 - 2\cos\theta$. Be sure that your graph illustrates all of the main features of the graph. Find (exactly!) the points at which the tangent line to the curve is vertical, and where it is horizontal.

4. Find the area between $r = 2\sin\theta$ and $r = 2\cos\theta$.

5. What is the projection of one vector onto another vector? Use projections to find the distance from the plane $3x - y + 2z = 6$ to the point $(1,0,-4)$.

6. Find the velocity, speed, acceleration, tangent and curvature of $\vec{r}(t) = \langle 2t, t^2, -\frac{1}{3}t^3 \rangle$.

7. Find an equation of the tangent plane to the hyperboloid $z^2 - 2x^2 - 2y^2 = 12$ at the point $(1,-1,4)$.

8. The simplest thermodynamic equation of state is the ideal gas law, $PV = \gamma T$ for some constant γ. Show that

$$\left(\frac{\partial P}{\partial V}\right)\left(\frac{\partial V}{\partial T}\right)\left(\frac{\partial T}{\partial P}\right) = -1.$$

   A more general equation of state would be of the form $F(P,V,T) = 0$ for some function F. Show that the previous statement still holds under this more general condition.

9. Find the local maximum and minimum values and saddle points of $f(x,y) = x^4 + y^4 - 4xy + 1$.

10. Use Lagrange multipliers to find the area of the largest rectangle with edges parallel to the axes that can be inscribed in the ellipse $x^2 + 4y^2 = 36$. Prove your result.

11. What is the gradient of a function? Suppose that the temperature in a region of space is given by $T(x,y,z) = \dfrac{80}{1 + x^2 + 2y^2 + 3z^2}$. In what direction does T increase fastest at $\langle 1,-1,2 \rangle$? What is the maximum rate of increase?

12. Give an informal and a precise definition of the limit $\lim_{(x,y)\to(a,b)} f(x,y)$. If $\lim_{x\to 0} f(x,0) = L$, and $\lim_{y\to 0} f(0,y) = L$ can we conclude that $\lim_{(x,y)\to(0,0)} f(x,y) = L$? Explain your answer.

13. Find the (exact) volume of the region bounded above by $z = \sqrt{16 - x^2 - y^2}$, and below by the *xy*-plane, within the cylinder $x^2 + y^2 = 9$.

14. Use a Jacobian change of variables to evaluate exactly $\iint_R \sin\left(\frac{x^2}{9} + \frac{y^2}{16}\right) dA$ where R is the region in the first quadrant inside the ellipse $\frac{x^2}{9} + \frac{y^2}{16} = \pi$.

15. Find the exact area of that part of the paraboloid $z = x^2 + y^2$ that lies inside the sphere $x^2 + y^2 + z^2 = 4z$.

16. Find the (exact) center of mass of the solid of uniform density bounded below by the upper nappe of the cone $z^2 = x^2 + y^2$, and above by the sphere $x^2 + y^2 + z^2 = a^2$.

17. Let C be the circle in the plane of radius 3 centered at the origin oriented counterclockwise, and let $\vec{F} = \left\langle x^2 + y^2, 2xy \right\rangle$. Calculate $\int_C \vec{F} \cdot d\vec{r}$ in two ways: directly and using Green's theorem.

18. Show that $\vec{F} = \left\langle y^2, 2xy + e^{3z}, 3ye^{3z} \right\rangle$ is conservative and find the potential. Use the fundamental theorem of line integrals to evaluate $\int_C \vec{F} \cdot d\vec{r}$ where C is portion of the semicubical parabola $\vec{r}(t) = \left\langle t, t^2, t^3 \right\rangle$ between $t = 0$ and $t = 2$.