Abstract

This paper considers the computational issue of the optimal stopping problem for the stochastic functional differential equation treated in [4]. The finite difference method developed by Barles and Souganidis [2] is used to obtain a numerical approximation for the viscosity solution of the infinite dimensional Hamilton-Jacobi-Bellman variational inequality (HJBJVI) associated with the optimal stopping problem.

KEYWORDS: optimal stopping, stochastic control, stochastic functional differential equations, finite difference.

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1 Introduction

Optimal stopping problems over a finite or an infinite time horizon for Itô’s diffusion processes arise in many areas of science, engineering, and finance (see Øksendal [21], Shiryaev [22], Karazas and Shreve [10] and references contained therein). The value function of these problems are normally expressed as a generalized solution of a variational inequality that involves a second order parabolic partial differential equation.

In the authors’ recent paper [4], an optimal stopping problem for a non-linear system with delay was considered. The value function was shown to be a unique viscosity solution of a HJB type variational inequality. However, no numerical results were given there. This paper gives a numerical method for the optimal stopping problems studied in [4]. The method we used here is the finite difference method which was introduced in Barles and Souganidis [2]. Similar method was used to deal with the general stochastic control problems with delay in the authors’ recent work [5]. Recently, Kushner [12] gives some numerical approximation results for general stochastic control problems for a stochastic functional differential equation using Markov Chain approximation method, which is entirely different from ours in the techniques and control problem considered.

This paper is organized as follows. In section 2, the formulation of the optimal stopping problem is given and the main result obtained in [4] is restated. In section 3, the numerical approximation of the viscosity solution of the infinite dimensional HJB variational inequality, along the line of finite difference method developed by Barles and Souganidis [20] is obtained and the convergence results are proved.

2 Problem Formulation

Let \( r > 0 \) be a fixed constant, and let \( J = [-r, 0] \) denote the duration of the bounded memory of the stochastic functional differential equations considered in this paper. For the sake of simplicity, denote \( C(J; \mathbb{R}^n) \), the space of continuous functions \( \phi : J \to \mathbb{R}^n \), by \( C \). Note that \( C \) is a real separable Banach space under the supremum-norm defined by

\[
\|\phi\| = \sup_{t \in J} |\phi(t)|, \quad \phi \in C
\]
where $| \cdot |$ is the Euclidean norm in $\mathbb{R}^n$.

We denote the inner product in $L^2(J, \mathbb{R}^n)$ by $(\cdot | \cdot)$ and the inner product in $\mathbb{R}^n$ by $\langle \cdot, \cdot \rangle$. Given $\phi$ and $\psi$ in $C$, the inner product $(\cdot | \cdot)$ and the norm $\| \cdot \|_2$ for $L^2(J, \mathbb{R}^n)$ are defined by

$$(\phi | \psi) = \int_{-r}^{0} \langle \phi(s), \psi(s) \rangle ds, \quad \text{and} \quad \| \phi \|_2 = (\phi | \phi)^{\frac{1}{2}}.$$  

Note that the space $C$ can be continuously embedded into $L^2(J; \mathbb{R}^n)$.

**Convention 2.1** For the rest of the paper, we use the following conventional notation for functional differential equations (see Hale [9]):

If $\psi \in C([-r, \infty); \mathbb{R}^n)$ and $t \in \mathbb{R}_+$, let $\psi_t \in C$ be defined by

$$\psi_t(\theta) = \psi(t + \theta), \quad \forall \theta \in J.$$  

Let $\{W(t), t \geq 0\}$ be an $m$-dimensional standard Brownian motion defined on a complete filtered probability space $(\Omega, \mathcal{F}, P; \mathcal{F})$, where $\mathcal{F} = \{\mathcal{F}(t), t \geq 0\}$ is the $P$-augmentation of the natural filtration $\{\mathcal{F}^W(t), t \geq 0\}$ generated by the Brownian motion $\{W(t), t \geq 0\}$, i.e., if $t \geq 0$,

$$\mathcal{F}^W(t) = \sigma\{W(s), 0 \leq s \leq t\}$$

and

$$\mathcal{F}(t) = \mathcal{F}^W(t) \vee \{A \subset \Omega | \exists B \in \mathcal{F} \text{ such that } A \subset B \text{ and } P(B) = 0\}$$

where the operator $\vee$ denotes that $\mathcal{F}(t)$ is the smallest $\sigma$-algebra such that $\mathcal{F}^W(t) \subset \mathcal{F}(t)$ and $\{A \subset \Omega | \exists B \in \mathcal{F} \text{ such that } A \subset B \text{ and } P(B) = 0\} \subset \mathcal{F}(t)$.

Let $T > 0$ and $t \in [0, T]$. Consider the following system of stochastic functional differential equations:

$$dX(s) = f(s, X_s)ds + g(s, X_s)dW(s), \quad s \in [t, T];$$

(1)

with the initial function $X_t = \psi_t$, where $\psi_t$ is a given $C$-valued random variable that is $\mathcal{F}(t)$-measurable. Here, $f : [0, T] \times C \to \mathbb{R}^n$ and $g : [0, T] \times C \to \mathbb{R}^{n \times m}$ are given deterministic functions.
Let $L^2(\Omega, C)$ be the space of $C$-valued random variables $\Xi : \Omega \to C$ such that
$$\|\Xi\|_{L^2} = \left\{ \int_{\Omega} \|\Xi(\omega)\|^2 dP(\omega) \right\}^{\frac{1}{2}} < \infty.$$ Let $L^2(\Omega, C; \mathcal{F}(t))$ be those $\Xi \in L^2(\Omega, C)$ which are $\mathcal{F}(t)$-measurable.

**Definition 2.2** A process $\{X(s; t, \psi_t), s \in [t-r, T]\}$ is said to be a (strong) solution of (1) on the interval $[t-r, T]$ and through the initial datum $(t, \psi_t) \in \mathbb{R}_+ \times L^2(\Omega, C; \mathcal{F}(t))$ if it satisfies the following conditions:

1. $X_t(\theta; t, \psi_t) = \psi_t(\theta), \ \forall \theta \in [-r, 0]$;
2. $X(s; t, \psi_t)$ is $\mathcal{F}(s)$-measurable for each $s \in [t, T]$;
3. The process $\{X(s; t, \psi_t), s \in [t, T]\}$ is continuous and it satisfies the following stochastic integral equation $P$-a.s.

$$X(s) = \psi_t(0) + \int_t^s f(\lambda, X_\lambda) d\lambda + \int_t^s g(\lambda, X_\lambda) dW(\lambda), \ s \in [t, T]. \ (2)$$

In addition, the solution process $\{X(s; t, \psi_t), s \in [t-r, T]\}$ is said to be (strongly) unique if $\tilde{X}(s; t, \psi_t), s \in [t-r, T]$ is also a solution of (1) on $[t-r, T]$ and through the same initial datum $(t, \psi_t)$, then
$$P\{X(s; t, \psi_t) = \tilde{X}(s; t, \psi_t), \forall s \in [t, T]\} = 1.$$

In this paper, we assume that the functions $f : [0, T] \times C \to \mathbb{R}^n$, and $g : [0, T] \times C \to \mathbb{R}^{n \times m}$ are continuous functions and satisfy the following conditions (Assumption 2.3 & 2.4) to ensure the existence and uniqueness of a global (strong) solution $\{X(s; t, \psi_t), s \in [t-r, T]\}$ for each $(t, \psi_t) \in [0, T] \times L^2(\Omega, C; \mathcal{F}(t))$. (See Mohammed [19, 20].)

**Assumption 2.3** There exists a constant $K > 0$ such that
$$|f(t, \varphi) - f(t, \phi)| + |g(t, \varphi) - g(t, \phi)| \leq K\|\varphi - \phi\|, \ \forall (t, \varphi), (t, \phi) \in [0, T] \times C.$$ **Assumption 2.4** There exists a constant $K > 0$ such that
$$|f(t, \phi)| + |g(t, \phi)| \leq K(1 + \|\phi\|), \ \forall (t, \phi) \in [0, T] \times C.$$
Let \( \{X(s; t, \psi_t), s \in [t, T]\} \) be the solution of (1) through the initial datum \((t, \psi_t) \in [0, T] \times \mathbb{C}\). We consider the corresponding \(\mathbb{C}\)-valued process \(\{X_s(t, \psi_t), s \in [t, T]\}\) defined by
\[
X_s(\theta; t, \psi_t) = X(s + \theta; t, \psi_t), \quad \theta \in J. \tag{3}
\]
Let \(G(t) = \{\mathcal{G}(t, s), s \in [t, T]\}\) be the filtration defined by
\[
\mathcal{G}(t, s) = \sigma(X(\lambda; t, \psi_t), \lambda \in [t, s]).
\]
Note that, it can be shown that for each \(s \in [t, T]\),
\[
\mathcal{G}(t, s) = \sigma(X_\lambda(t, \psi_t), \lambda \in [t, s]).
\]
This is due to the sample paths continuity of the process \(\{X(s; t, \psi_t), s \in [t, T]\}\).

One can then establish the following Markov property (see Mohammed [19], [20]):

**Theorem 2.5** Let Assumptions 2.3 and 2.4 hold. Then the corresponding \(\mathbb{C}\)-valued solution process of (1) describes a \(\mathbb{C}\)-valued Markov process in the following sense:

For any \((t, \psi_t) \in [0, T] \times L^2(\Omega, \mathbb{C})\), the Markovian property
\[
P\{X_s(t, \psi_t) \in B | \mathcal{G}(t, \alpha)\} = P\{X_s(t, \psi_t) \in B | X_\alpha(t, \psi_t)\} \equiv p(\alpha, X_\alpha(t, \psi_t), s, B)
\]
holds a.s. for \(t \leq \alpha \leq s\) and \(B \in \mathcal{B}(\mathbb{C})\), where \(\mathcal{B}(\mathbb{C})\) is the Borel \(\sigma\)-algebra of subsets of \(\mathbb{C}\).

In the above, the function \(p: [t, T] \times \mathbb{C} \times [t, T] \times \mathcal{B}(\mathbb{C}) \to [0, 1]\) denotes the transition probabilities of the \(\mathbb{C}\)-valued Markov process \(\{X_s(t, \psi_t), s \in [t, T]\}\).

Let \(\mathcal{T}_T^T\) be the collection of all \(\mathcal{G}(t)\)-stopping times \(\tau: \Omega \to [0, \infty]\) such that \(t \leq \tau \leq T\) a.s. We write \(\mathcal{T}_0^T = \mathcal{T}\) when \(t = 0\) and \(T = \infty\). For each \(\tau \in \mathcal{T}_T^T\), let the sub-\(\sigma\)-algebra \(\mathcal{G}(t, \tau)\) of \(\mathcal{F}\) be defined by
\[
\mathcal{G}(t, \tau) = \{A \in \mathcal{F} \mid A \cap \{t \leq \tau \leq s\} \in \mathcal{G}(t, s) \forall s \in [t, T]\}.
\]

With a little bit more effort, one can also shows that the corresponding \(\mathbb{C}\)-valued process of (1) is also a strong Markov process in \(\mathbb{C}\). That is
\[
P\{X_s(t, \psi_t) \in B | \mathcal{G}(t, \tau)\} = P\{X_s(t, \psi_t) \in B | X_\tau(t, \psi_t)\} \equiv p(\tau, X_\tau(t, \psi_t), s, B)
\]
holds a.s. for all $\tau \in T_t^T$ and $B \in \mathcal{B}(C)$.

If the drift coefficient $f$ and the diffusion coefficient $g$ are time-independent, i.e., $f(t, \phi) \equiv f(\phi)$ and $g(t, \phi) \equiv g(\phi)$, then (1) reduces to the following autonomous system:

$$dX(s) = f(X_s)ds + g(X_s)dW(s). \quad (4)$$

In this case, we usually assume the initial datum $(t, \psi_t) = (0, \psi)$ and denote the solution process of (4) through $(0, \psi)$ and on the interval $[-r, T]$ by $\{X(s; \psi), s \in [-r, T]\}$. Then the corresponding $C$-valued process $\{X_s(\psi), s \in [-r, T]\}$ of (4) is a strong Markov process with time-homogeneous probability transition probabilities $p(\psi, s, B) \equiv p(0, \psi, s, B) = p(t, \psi, t + s, B)$ for all $s, t \geq 0$, $\psi \in C$, and $B \in \mathcal{B}(C)$.

Assume that $L$ and $\Psi$ are two Lipschitz continuous real-valued functions on $[0, T] \times C$ with at most linear growth in $L^2(J; \mathbb{R}^n)$. In other words, there exist constants $K_1, K_2$ such that

$$|L(t, \phi)| \leq K_1(1 + \|\phi\|_2), \quad \text{and} \quad |\Psi(t, \phi)| \leq K_2(1 + \|\phi\|_2),$$

for all $(t, \phi) \in [0, T] \times C$.

Our objective is to find an optimal stopping time $\tau^* \in T_t^T$ that maximizes the following expected cost functional:

$$J(\tau; t, \psi) = E \left[ \int_t^\tau e^{-\rho(s-t)} L(s, X_s)ds + e^{-\rho(\tau-t)}\Psi(\tau, X_\tau) \right], \quad (5)$$

where $\rho > 0$ denotes a discount factor. In this case, the value function $V : [0, T] \times C \to \mathbb{R}$ is defined to be

$$V(t, \psi) = \sup_{\tau \in T_t^T} J(\tau; t, \psi). \quad (6)$$

Before we give the HJBVI satisfied by $V(t, \psi)$, we need to introduce some spaces and operators. Let $C^*$ and $C^\dagger$ be the space of bounded linear functionals $\Phi : C \to \mathbb{R}$ and bounded bilinear functionals $\tilde{\Phi} : C \times C \to \mathbb{R}$, of the space $C$, respectively. They are equipped with the operator norms which will be, respectively, denoted by $\| \cdot \|^*$ and $\| \cdot \|^\dagger$. 

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Let $B = \{v1_{(0)} : v \in \mathbb{R}^n\}$, where $1_{(0)} : [-r, 0] \to \mathbb{R}$ is defined by

$$1_{(0)}(\theta) = \begin{cases} 0 & \text{for } \theta \in [-r, 0), \\ 1 & \text{for } \theta = 0. \end{cases}$$

We form the direct sum

$$C \oplus B = \{\phi + v1_{(0)} : \phi \in C, v \in \mathbb{R}^n\}$$

and equip it with the norm $\| \cdot \|$ defined by

$$\|\phi + v1_{(0)}\| = \sup_{\theta \in [-r, 0]} |\phi(\theta)| + |v|, \quad \phi \in C, v \in \mathbb{R}^n.$$  

Note that for each sufficiently smooth function $\Phi : C \to \mathbb{R}$, its first order Fréchet derivative (with respect to $\phi \in C$), $D\Phi(\varphi) \in C^*$, has a unique and continuous linear extension $D\Phi(\varphi) \in (C \oplus B)^*$. Similarly, its second order Fréchet derivative, $D^2\Phi(\varphi) \in C^\dagger$, has the unique and continuous linear extension $D^2\Phi(\varphi) \in (C \oplus B)^\dagger$, where $(C \oplus B)^*$ and $(C \oplus B)^\dagger$ are spaces of bounded linear and bilinear functionals of $C \oplus B$, respectively. (See Lemma (3.1) and Lemma (3.2) on pp 79-83 of Mohammed [19] for details).

For a Borel measurable function $\Phi : C \to \mathbb{R}$, we also define

$$S(\Phi)(\phi) = \lim_{h \to 0^+} \frac{1}{h} \left[ \Phi(\tilde{\phi}_h) - \Phi(\phi) \right]$$

for all $\phi \in C$, where $\tilde{\phi} : [-r, T] \to \mathbb{R}^n$ is an extension of $\phi$ defined by

$$\tilde{\phi}(t) = \begin{cases} \phi(t) & \text{if } t \in [-r, 0) \\ \phi(0) & \text{if } t \geq 0, \end{cases}$$

and again $\tilde{\phi}_t \in C$ is defined by

$$\tilde{\phi}_t(\theta) = \tilde{\phi}(t + \theta), \quad \theta \in [-r, 0].$$

Let $\hat{D}(S)$, the domain of the operator $S$, be the set of $\Phi : C \to \mathbb{R}$ such that the above limit exists for each $\phi \in C$.

Define $C_{lip}^{1,2}([0, T] \times C)$ as the space of functions $\Phi : [0, T] \times C \to \mathbb{R}$ such that $\frac{\partial \Phi}{\partial t} : [0, T] \times C \to \mathbb{R}$ and $D^2\Phi : [0, T] \times C \to C^\dagger$ exist and are continuous and satisfy the following Lipschitz condition:

$$\|D^2\Phi(t, \phi) - D^2\Phi(t, \varphi)\| \leq K \|\phi - \varphi\|, \quad \forall t \in [0, T], \forall \phi, \varphi \in C,$$
where $K > 0$ is a constant.

Let $\mathcal{D}(S)$ be the collection of those $\Phi : [0,T] \times C \to \mathbb{R}$ such that $\Phi(t, \cdot) \in \hat{\mathcal{D}}(S)$ for each $t \in [0, T]$.

The following result has been proved in [4].

**Theorem 2.6** The value function $V$ is the unique viscosity solution of the HJBVI

$$\max \left\{ \Psi(t, \psi) - V(t, \psi), \frac{\partial V}{\partial t}(t, \psi) + \mathcal{A}V(t, \psi) + L(t, \psi) - \rho V(t, \psi) \right\} = 0,$$

for all $(t, \psi) \in [0, T) \times C$. \hfill (7)

and $V(T, \psi) = \Psi(T, \psi)$ for all $\psi \in C$, where $\mathcal{A}$ is given by

$$\mathcal{A}V(t, \psi) = S(V)(t, \psi) + DV(t, \psi)(f(t, \psi)1_{\{0\}})$$

$$+ \frac{1}{2} \sum_{i=1}^{m} D^2V(t, \psi)(g(t, \psi)e_i1_{\{0\}}, g(t, \psi)e_i1_{\{0\}}), \hfill (8)$$

where $e_i$ is the $i$-th unit vector of the standard basis in $\mathbb{R}^m$.

### 3 Finite Difference Approximation

In this section, we consider an explicit finite difference scheme and show that it converges to the unique viscosity solution of equation (7). We will use a method introduced by Barles and Souganidis [2].

To obtain the existence results of the finite difference equation which will be given later, we will use the Banach fixed point theorem. Therefore, it will be more convenient to consider the bounded functionals. For this reason, we will use a truncated optimal stopping problem as the follows.

Given a positive integer $M$, we consider the following truncated optimal stopping problem with value function $V_M : [0, T] \times C \to \mathbb{R}$

$$V_M(t, \psi) = \sup_{\tau \in T^+_t} \mathbb{E} \left[ \int_t^\tau e^{-\rho(s-t)}(L(s, X_s) \wedge M) ds \
+ e^{-\rho(\tau-t)}(\Psi(\tau, X_\tau) \wedge M) \right], \hfill (9)$$
where $a \land b$ is defined by $a \land b = \min \{a, b\}$ for all $a, b \in \mathbb{R}$.

The corresponding HJBVI is given by

$$\min \left\{ V_M(t, \psi) - (\Psi(t, \psi) \land M), \rho V_M(t, \psi) - \frac{\partial V_M}{\partial t}(t, \psi) - AV_M(t, \psi) - (L(t, \psi) \land M) \right\} = 0, \quad \forall (t, \psi) \in [0, T] \times C, \quad (10)$$

and $V_M(T, \psi) = \min(\Psi(T, \psi), M)$, $\forall \psi \in C$.

Similarly as in [4], it can be shown that the value function $V_M$ is the unique viscosity solution of the equation (10).

Moreover, it is easy to see that $V_M \to V$ as $M \to \infty$. In view of these, we need only find the numerical solution for $V_M$.

Let $\varepsilon$ with $0 < \varepsilon < 1$ be the stepsize for variables $\psi$ and $\eta$, where $0 < \eta < 1$ is the stepsize for $t$. We consider the finite difference operators $\Delta_\eta$, $\Delta_\varepsilon$ and $\Delta^2_\varepsilon$ defined by

$$\Delta_\eta W(t, \psi) \equiv \frac{W(t + \eta, \psi) - W(t, \psi)}{\eta},$$

$$\Delta_\varepsilon W(t, \psi)(h + v \mathbf{1}_{\{0\}}) \equiv \frac{W(t, \psi + \varepsilon(h + v \mathbf{1}_{\{0\}})) - W(t, \psi)}{\varepsilon},$$

$$\Delta^2_\varepsilon W(t, \psi)(h + v \mathbf{1}_{\{0\}}, k + w \mathbf{1}_{\{0\}}) \equiv \frac{W(t, \psi + \varepsilon(h + v \mathbf{1}_{\{0\}})) - W(t, \psi)}{\varepsilon^2} + \frac{W(t, \psi - \varepsilon(k + w \mathbf{1}_{\{0\}})) - W(t, \psi)}{\varepsilon^2},$$

where $h, k \in C$ and $v, w \in \mathbb{R}^n$. Recall that

$$S(\Phi)(\phi) = \lim_{\varepsilon \to \Phi + \varepsilon} \frac{1}{\varepsilon} \left[ \Phi(\tilde{\phi}_\varepsilon) - \Phi(\phi) \right].$$

Therefore we define

$$S_\varepsilon(\Phi)(\phi) \equiv \frac{1}{\varepsilon} \left[ \Phi(\tilde{\phi}_\varepsilon) - \Phi(\phi) \right].$$

It is clear that $S_\varepsilon(\Phi)$ is an approximation of $S(\Phi)$ as $\varepsilon$ approaches 0.

Next we will show that $\Delta_\varepsilon W(t, \psi)$ and $\Delta^2_\varepsilon W(t, \psi)$ are the approximations of $\Delta W(t, \psi)$ and $\Delta^2 W(t, \psi)$ respectively.
Lemma 3.1 For any \( W : [0, T] \times C \rightarrow \mathbb{R}, W \in C^{1,2}([0, T] \times C) \) such that \( W \) can be smoothly extended on \( [0, T] \times (C \oplus B) \), we have

\[
\lim_{\varepsilon \to 0} \Delta_{\varepsilon} W(t, \psi)(h + v_1\{0\}) = DW(t, \psi)(h + v_1\{0\}),
\]

and

\[
\lim_{\varepsilon \to 0} \Delta_{\varepsilon}^2 W(t, \psi)(h + v_1\{0\}, k + w_1\{0\}) = D^2W(t, \psi)(h + v_1\{0\}, k + w_1\{0\}).
\]

Proof. Note that the function \( W \) can be extended from \([0, T] \times C\) to \([0, T] \times (C \oplus B)\). Let us denote by \( \tilde{W} \) the smooth extension of \( W \) to \([0, T] \times (C \oplus B)\).

It is clear that

\[
\lim_{\varepsilon \to 0} \Delta_{\varepsilon} W(t, \psi)(h + v_1\{0\}) = d_G \tilde{W}(t, \psi)(h + v_1\{0\})
\]

where \( d_G \tilde{W} \) denote the Gâteau derivative of \( \tilde{W} \) with respect to its second variable. And since \( \tilde{W} \) is smooth then the Gâteau derivative and the Fréchet derivative of \( \tilde{W} \) coincide. Moreover, they are the continuous extension of the \( DW \), the Fréchet derivative of \( W \). On the other hand, the uniqueness of the linear continuous extension give us the following:

\[
\lim_{\varepsilon \to 0} \Delta_{\varepsilon} W(t, \psi)(h + v_1\{0\}) = \lim_{\varepsilon \to 0} \Delta_{\varepsilon} \tilde{W}(t, \psi)(h + v_1\{0\}) = DW(t, \psi)(h + v_1\{0\}).
\]

Similar argument can be used for (12).

Let \( \varepsilon, \eta > 0 \), the corresponding discrete version of equation (10) is given by

\[
\min \left\{ V_M(t, \psi) - (\Psi(t, \psi) \wedge M), \right. \\
\rho V_M(t, \psi) - \frac{V_M(t + \eta, \psi) - V_M(t, \psi)}{\eta} - \frac{V_M(t, \tilde{\psi} \varepsilon) - V_M(t, \psi)}{\varepsilon} \\
- \frac{V_M(t, \psi + \varepsilon(f(t, \psi)1\{0\})) - V_M(t, \psi)}{\varepsilon} \\
- \frac{1}{2} \sum_{i=1}^{m} \left( \frac{V_M(t, \psi + \varepsilon(g(t, \psi)e_i1\{0\})) - V_M(t, \psi)}{\varepsilon^2} \right)
\]
\[ \begin{aligned} &+ \frac{V_M(t, \psi - \varepsilon(g(t, \psi)e_i1\{0}\)) - V_M(t, \psi)}{\varepsilon^2} \bigg) \\
&- (L(t, \psi) \wedge M) \bigg] = 0. \tag{14} \end{aligned} \]

Rearranging terms, we obtain

\[
\min \left[ V_M(t, \psi) - (\Psi(t, \psi) \wedge M), \right.
\]
\[
\left( \frac{2}{\varepsilon} + \frac{1}{\eta} + \frac{m}{\varepsilon^2} + \rho \right) V_M(t, \psi) - \frac{1}{\varepsilon} V_M(t, \tilde{\psi}_\varepsilon)
\]
\[
- \frac{V_M(t, \psi + \varepsilon(f(t, \psi)1\{0}\))}{\varepsilon} - \frac{V_M(t + \eta, \psi)}{\eta}
\]
\[
- \frac{1}{2} \sum_{i=1}^{m} \frac{V_M(t, \psi + \varepsilon(g(t, \psi)e_i1\{0\})) + V_M(t, \psi - \varepsilon(g(t, \psi)e_i1\{0\}))}{\varepsilon^2}
\]
\[
- (L(t, \psi) \wedge M) \bigg] = 0. \tag{15} \]

Let \( C([0, T] \times (\mathbb{C} \oplus \mathbb{B}))_b \) denote the space of bounded continuous functions \( W \) from \([0, T] \times (\mathbb{C} \oplus \mathbb{B})\) to \( \mathbb{R} \). Define a mapping \( S_M : (0, 1)^2 \times [0, T] \times \mathbb{C} \times \mathbb{R} \times C([0, T] \times (\mathbb{C} \oplus \mathbb{B}))_b \rightarrow \mathbb{R} \) as the following

\[
S_M(\varepsilon, \eta, t, \psi, x, W) \equiv \varepsilon \min \left[ x - (\Psi(t, \psi) \wedge M), \right.
\]
\[
\left( \frac{2}{\varepsilon} + \frac{1}{\eta} + \frac{m}{\varepsilon^2} + \rho \right)x - \frac{1}{\varepsilon} W(t, \tilde{\psi}_\varepsilon)
\]
\[
- \frac{W(t, \psi + \varepsilon(f(t, \psi)1\{0\}))}{\varepsilon} - \frac{W(t + \eta, \psi)}{\eta}
\]
\[
- \frac{1}{2} \sum_{i=1}^{m} \frac{W(t, \psi + \varepsilon(g(t, \psi)e_i1\{0\})) + W(t, \psi - \varepsilon(g(t, \psi)e_i1\{0\}))}{\varepsilon^2}
\]
\[
- (L(t, \psi) \wedge M) \bigg]. \tag{16} \]

Then, (14) is equivalent to

\[
S_M(\varepsilon, \eta, t, \psi, V_M(t, \psi), V_M) = 0.
\]

Moreover, note that the coefficients of all the terms that involve \( W \) in \( S_M \) are negative. This implies that \( S_M \) is monotone, i.e., for all \( W_1, W_2 \in \mathbb{R} \), if \( W_1 \leq W_2 \) then \( S_M(\varepsilon, \eta, t, \psi, W_1, V_M) \leq S_M(\varepsilon, \eta, t, \psi, W_2, V_M) \).
\( C([0, T] \times (C \oplus B))_b, \varepsilon, \eta \in (0, 1), t \in [0, T], \psi \in C, \) and \( x \in \mathbb{R}, \) we have
\[
S_M(\varepsilon, \eta, t, \psi, x, W_1) \leq S_M(\varepsilon, \eta, t, \psi, x, W_2) \text{ whenever } W_1 \geq W_2. \tag{17}
\]

**Definition 3.2** The scheme \( S_M \) is said to be consistent if, for every \( t \in [0, T], \psi \in C \oplus B, \) and for every test function \( W \in C^{1,2}([0, T] \times (C \oplus B))_b, \)
\[
\min \left\{ W(t, \psi) - (\Psi(t, \psi) \land M), \right. \\
\left. \rho W(t, \psi) - \frac{\partial W}{\partial t}(t, \psi) - AW(t, \psi) - (L(t, \psi) \land M) \right\}
= \lim_{(\tau, \phi) \to (t, \psi), \varepsilon, \eta \to 0, \xi \to 0} S_M(\varepsilon, \eta, \tau, \phi, W(\tau, \phi) + \xi, W + \xi).
\]

We have the following result:

**Lemma 3.3** The scheme \( S_M \) defined by (16) is consistent.

**Proof.** Let \( W \in C^{1,2}([0, T] \times (C \oplus B))_b \cap D(S). \) We write
\[
S_M(\varepsilon, \eta, \tau, \phi, W(\tau, \phi) + \xi, W + \xi)
= \min_{\varepsilon} \left\{ W(\tau, \phi) - (\Psi(\tau, \phi) \land M), \right.
\frac{2}{\varepsilon} + \frac{1}{\eta} + \frac{m}{\varepsilon^2} + \rho(W(\tau, \phi) + \xi) - \frac{W(\tau + \eta, \phi) + \xi}{\eta} - \frac{1}{\varepsilon}(W(\tau, \tilde{\phi}_e) + \xi)
- \frac{1}{2} \sum_{i=1}^{m} W(\tau, \phi + \varepsilon(g(\tau, \phi)e_i1_{\{0\}})) + 2\xi + W(\tau, \phi - \varepsilon(g(\tau, \phi)e_i1_{\{0\}})) \right. \\
\left. \left( L(\tau, \phi) \land M \right) \right\}
= \min_{\varepsilon} \left\{ W(\tau, \phi) - (\Psi(\tau, \phi) \land M), \right.
\frac{\rho(W(\tau, \phi) + \xi) - W(\tau + \eta, \phi) - W(\tau, \phi)}{\eta} - \frac{W(\tau, \tilde{\phi}_e) - W_M(\tau, \phi)}{\varepsilon}
- \frac{W(\tau, \phi + \varepsilon(f(\tau, \phi)1_{\{0\}})) - W(\tau, \phi)}{\varepsilon}
- \frac{1}{2} \sum_{i=1}^{m} \left( W(\tau, \phi + \varepsilon(g(\tau, \phi)e_i1_{\{0\}})) - W(\tau, \phi) \right) \right. \\
\left. \left( L(\tau, \phi) \land M \right) \right\}.
\[
-W(\tau, \phi) + \frac{W(\tau, \phi - \varepsilon(g(\tau, \phi)e_1\{0\})) - W(\tau, \phi)}{\varepsilon^2} \right)
\]

\[
-(L(\tau, \phi) \wedge M) \right)
\]

= \min \left\{ \left( W(\tau, \phi) + \xi \right) - (\Psi(\tau, \phi) \wedge M), \right.
\rho(W(\tau, \phi) + \xi) - \Delta_{\eta}W(\tau, \phi) - S_{\varepsilon}(W(\tau, \cdot))(\phi)
\Delta_{\varepsilon}W(\tau, \phi)(f(\tau, \phi)e_1\{0\}) - \frac{1}{2} \sum_{i=1}^{m} \Delta_{\varepsilon}^{2}W(\tau, \phi)(g(\tau, \phi)e_i\{0\})
\left. -(L(\tau, \phi) \wedge M) \right\} \right\}
\]

Using Lemma 3.1, and sending \( \xi \to 0, \tau \to t, \phi \to \psi, \varepsilon, \eta \to 0 \) in (18), we deduce

\[
\min \left\{ W(t, \psi) - \Psi(t, \psi) \wedge M, \right.
\rho W(t, \psi) - \frac{\partial W}{\partial t}(t, \psi) - A W(t, \psi) - (L(t, \psi) \wedge M) \right. \}
\]

\[
= \lim_{(\tau, \phi) \to (t, \psi), \varepsilon, \eta \to 0, \xi \to 0} \frac{S_M(\varepsilon, \eta, \tau, \phi, W(t, \psi) + \xi, W + \xi)}{\varepsilon}.
\]

This completes the proof. \( \Box \)

Next, we will show that the equation

\[
S_M(\varepsilon, \eta, t, \psi, W(t, \psi), W) = 0 \tag{19}
\]

has a solution. Using (15), we see that the equation \( S_M(\varepsilon, \eta, t, \psi, W(t, \psi), W) = 0 \) is equivalent to the equation

\[
W(t, \psi) = \max \left[ \left( \Psi(t, \psi) \wedge M, \right. \right.
\frac{1}{\varepsilon + \frac{1}{\eta} + \frac{m}{\varepsilon^2} + \rho} \left( \frac{W(t, \psi + \varepsilon(f(t, \psi)e_1\{0\}))}{\varepsilon} \right)
\left. - \frac{1}{2} \sum_{i=1}^{m} \frac{W(t, \psi + \varepsilon(g(t, \psi)e_i\{0\})) + W(t, \psi - \varepsilon(g(t, \psi)e_i\{0\}))}{\varepsilon^2} \right)
\left. - \frac{1}{\varepsilon} W(t, \psi \varepsilon) - \frac{W(t + \eta, \psi)}{\eta} - (L(t, \psi) \wedge M) \right] \right]. \tag{20}
\]
We define an operator $T_{\epsilon,\eta}$ on $C_b([0, T] \times (C \oplus B))$ as follows,

$$
T_{\epsilon,\eta} W(t, \psi) 
\equiv 
\max \left[ \left( \frac{1}{\epsilon} + \frac{1}{\eta} + \frac{m}{\epsilon^2} + \rho \right) \left( \frac{W(t, \psi + \epsilon(f(t, \psi)e_1(0)))}{\epsilon} \right) + W(t, \psi - \epsilon(g(t, \psi)e_1(0))) 
- \frac{1}{\epsilon} W(t, \tilde{\psi}_\epsilon) - \frac{1}{\eta} (L(t, \psi) \wedge M) \right] + \frac{1}{\epsilon^2} \sum_{i=1}^{m} \frac{W(t, \psi + \epsilon(g(t, \psi,e_i)1_{(0)})) + W(t, \psi - \epsilon(g(t, \psi,e_i)1_{(0)}))}{\epsilon^2} - \frac{1}{\epsilon} W(t, \tilde{\psi}_\epsilon) - \frac{1}{\eta} (L(t, \psi) \wedge M) \right],
$$

(21)

If we can show that $T_{\epsilon,\eta}$ is a contraction map, then we can obtain the existence result for (19).

**Lemma 3.4** For each $\epsilon > 0$ and $\eta > 0$, $T_{\epsilon,\eta}$ is a contraction map.

**Proof.** To prove that $T_{\epsilon,\eta}$ is a contraction map, we need to show that there exists a constant $0 < \beta < 1$ such that

$$
\|T_{\epsilon,\eta} W_1 - T_{\epsilon,\eta} W_2\| \leq \beta \|W_1 - W_2\| \quad \text{for all } W_1, W_2 \in C([0, T] \times (C \oplus B))_b,
$$

where $\| \cdot \|$ is the supremum norm. Let us define $c_{\epsilon,\eta}$ by

$$
c_{\epsilon,\eta} \equiv \frac{2}{\epsilon} + \frac{1}{\eta} + \frac{m}{\epsilon^2} + \rho.
$$

Now we have

$$
|T_{\epsilon,\eta} W_1(t, \psi) - T_{\epsilon,\eta} W_2(t, \psi)|
\leq \max \left[ \left( \frac{1}{\epsilon} W_1(t, \psi + \epsilon(f(t, \psi,e_i)e_1(0))) + \frac{W_1(t, \psi - \epsilon(g(t, \psi,e_i)e_1(0)))}{\epsilon} \right) + \frac{1}{\epsilon^2} \sum_{i=1}^{m} \frac{W_1(t, \psi + \epsilon(g(t, \psi,e_i)e_1(0))) + W_1(t, \psi - \epsilon(g(t, \psi,e_i)e_1(0)))}{\epsilon^2} - \frac{1}{\epsilon} W_2(t, \tilde{\psi}_\epsilon) - \frac{1}{\eta} (L(t, \psi) \wedge M) \right] + \frac{1}{\epsilon^2} \sum_{i=1}^{m} \frac{W_2(t, \psi + \epsilon(g(t, \psi,e_i)e_1(0))) + W_2(t, \psi - \epsilon(g(t, \psi,e_i)e_1(0)))}{\epsilon^2} - \frac{1}{\epsilon} W_2(t, \tilde{\psi}_\epsilon) - \frac{1}{\eta} (L(t, \psi) \wedge M) \right].
$$

Noting that $\| \cdot \|$ denotes the supremum norm, the above inequality implies that, for all $t, \psi$,

$$
|T_{\epsilon,\eta} W_1(t, \psi) - T_{\epsilon,\eta} W_2(t, \psi)| \leq \left[ \frac{2 + \frac{1}{\epsilon} + \frac{m}{\epsilon^2}}{c_{\epsilon,\eta}} \right] \|W_1 - W_2\|.
$$
In addition, note that
\[
\frac{2}{c_{\varepsilon,\eta}} + \frac{1}{\eta} + \frac{m}{c_{\varepsilon,\eta}^2} = \frac{2}{c_{\varepsilon,\eta}} + \frac{1}{\eta} + \frac{m}{c_{\varepsilon,\eta}^2} + \rho < 1.
\]

Take
\[
\beta \equiv \frac{2}{c_{\varepsilon,\eta}} + \frac{1}{\eta} + \frac{m}{c_{\varepsilon,\eta}^2}.
\]

Then we have that \(0 < \beta < 1\) and
\[
\|T_{\varepsilon,\eta}W_1 - T_{\varepsilon,\eta}W_2\| \leq \beta\|W_1 - W_2\|.
\]

This completes the proof. \(\square\)

**Definition 3.5** The scheme \(S_M\) is said to be **stable** if for every \(\varepsilon, \eta \in (0,1)\), there exists a bounded solution \(W_{\varepsilon,\eta} \in C([0,T] \times (\mathbb{C} \oplus \mathbb{B}))_b\) to the equation
\[
S_M(\varepsilon, \eta, t, \psi, W(t, \psi), W) = 0,
\]
with the bound independent of \(\varepsilon\) and \(\eta\).

We have the following result:

**Lemma 3.6** The scheme \(S_M\) defined by (16) is stable.

**Proof.** By the Banach fixed point theorem, the strict contraction \(T_{\varepsilon,\eta}\) defined by (21) has a unique fixed point that we denote by \(W^M_{\varepsilon,\eta}\). From the definition of \(T_{\varepsilon,\eta}\), it is easy to see that \(W^M_{\varepsilon,\eta}\) is actually the solution of the equations
\[
S_M(\varepsilon, \eta, t, \psi, W(t, \psi), W) = 0,
\]
where the scheme \(S_M\) is defined by (16). Next we need to show that the solution \(W^M_{\varepsilon,\eta}\) has a bound which is independent of \(\varepsilon, \eta\).

Given any function \(W_0 \in C([0,T] \times (\mathbb{C} \oplus \mathbb{B}))_b\), we construct a sequence as follows, \(W_{n+1} = T_{\varepsilon,\eta}W_n\) for \(n \geq 0\). It is clear that
\[
\lim_{n \to \infty} W_n = W^M_{\varepsilon,\eta}.
\]
Moreover, we have

\[
W_{n+1}(t, \psi) = \max \left[ \left( \Psi(t, \psi) \land M \right), \frac{1}{\varepsilon + 1} + \frac{m}{\varepsilon^2} + \rho \left( \frac{W_n(t, \psi + \varepsilon(f(t, \psi) \mathbf{1}_{\{0\}}))}{\varepsilon} \right. \right.
\]
\[
- \frac{1}{2} \sum_{i=1}^{m} \frac{W_n(t, \psi + \varepsilon(g(t, \psi, e_i \mathbf{1}_{\{0\}}))) + W_n(t, \psi - \varepsilon(g(t, \psi, e_i \mathbf{1}_{\{0\}})))}{\varepsilon^2}
\]
\[
- \frac{1}{\varepsilon} W_n(t, \tilde{\psi}) - \frac{W_n(t + \eta, \psi)}{\eta} - (L(t, \psi) \land M) \right] \right] \tag{23}
\]

By virtue of

\[
0 < \frac{c_{\varepsilon, \eta} - \rho}{c_{\varepsilon, \eta}} < 1,
\]

we can get that

\[
\| W_{n+1} \| \leq \max \left[ M, \frac{c_{\varepsilon, \eta} - \rho}{c_{\varepsilon, \eta}} \| W_n \| + \frac{1}{c_{\varepsilon, \eta}} M \right]. \tag{24}
\]

From (24), we deduce that

\[
\| W_{n+1} \| \leq \max \left[ M, \left( \frac{c_{\varepsilon, \eta} - \rho}{c_{\varepsilon, \eta}} \right)^{n+1} \| W_0 \| + \frac{M}{c_{\varepsilon, \eta}} \sum_{i=0}^{n} \left( \frac{c_{\varepsilon, \eta} - \rho}{c_{\varepsilon, \eta}} \right)^i \right].
\]

Taking the limit as \( n \to \infty \), we obtain

\[
\| W^M_{\varepsilon, \eta} \| \leq \frac{M}{c_{\varepsilon, \eta}} \cdot \frac{1}{1 - \frac{c_{\varepsilon, \eta} - \rho}{c_{\varepsilon, \eta}}} = \frac{M}{\rho}.
\]

This implies the stability of the scheme \( S_M \).

Given the results of Lemma 3.3 and Lemma 3.6, now we are ready to show the main result of this paper:

**Theorem 3.7** Let \( W^M_{\varepsilon, \eta} \) denote the solution to (22). Then, as \((\varepsilon, \eta) \to 0\), the sequence \( W^M_{\varepsilon, \eta} \) converges uniformly on \([0, T] \times C\) to the unique viscosity solution \( V_M \) of (10).
Proof. Define
\[
W^*_M(t, \psi) = \limsup_{\tau \to t, \phi \to \psi, \epsilon, \eta \to 0} W^M(\tau, \phi), \\
W_\ast M(t, \psi) = \liminf_{\tau \to t, \phi \to \psi, \epsilon, \eta \to 0} W^M(\tau, \phi).
\] (25)

We claim that $W^*_M$ and $W_\ast M$ are subsolution and supersolutions of (10), respectively. To prove this claim, we only consider the case for $W^*_M$. The argument for that of $W_\ast M$ is similar.

To prove that $W^*_M$ is the subsolution of (10), we need to show:
\[
\min \left\{ \Gamma(t, \psi) - (\Psi(t, \psi) \wedge M), \rho \Gamma(t, \psi) - \frac{\partial \Gamma}{\partial t}(t, \psi) - S(\Gamma)(t, \psi) \\
- D\Gamma(t, \psi)(f(t, \psi)1_{\{0\}}) - \frac{1}{2} \sum_{i=1}^{m} D^2\Gamma(t, \psi)(g(t, \psi)e_i1_{\{0\}}, g(t, \psi)e_i1_{\{0\}}) \\
- (L(t, \psi) \wedge M) \right\} \leq 0,
\]
for any test function $\Gamma \in C^{1,2}_{lip}([0, T] \times (\mathbf{C} \oplus \mathbf{B})) \cap D(S)$ such that $(t, \psi)$ is a strictly local maximum of $W^*_M(\tau, \phi) - \Gamma(\tau, \phi)$. Without loss of generality, here we assume that $W^*_M \leq \Gamma$ and $W^*_M(t, \psi) = \Gamma(t, \psi)$ in a neighborhood $B((t, \psi), l)$ of $(t, \psi)$.

Moreover, by virtue of Lemma 3.6, we know that our scheme is stable, thus we can also assume that $\Gamma \geq 2 \sup_{\epsilon, \eta} \|W^M_{\epsilon, \eta}\|$ outside of the ball $B((t, \psi), l)$ where $l > 0$ satisfies
\[
W^*_M(\tau, \phi) - \Gamma(\tau, \phi) \leq 0 = W^*_M(t, \psi) - \Phi(t, \psi) \text{ for } (\tau, \phi) \in B((t, \psi), l).
\]

This implies that there exist sequences $\epsilon_n > 0$, $\eta_n > 0$, and $(\tau_n, \phi_n) \in [0, T] \times (\mathbf{C} \oplus \mathbf{B})$ such that as $n \to \infty$ we have
\[
\epsilon_n \to 0, \ \eta_n \to 0, \ \tau_n \to t, \ \phi_n \to \psi, \ W^M_{\epsilon_n, \eta_n}(\tau_n, \phi_n) \to W^*_M(t, \psi),
\]
and $(\tau_n, \phi_n)$ is a global maximum $W^M_{\epsilon_n, \eta_n} - \Gamma$,

where $W^M_{\epsilon_n, \eta_n}$ is the solution of the equation
\[
S_M(\epsilon_n, \eta_n, \tau_n, \phi_n, W(\tau_n, \phi_n), W) = 0,
\]
Denote $\alpha_n = W^M_{\epsilon_n, \eta_n}(\tau_n, \phi_n) - \Gamma(\tau_n, \phi_n)$. Obviosly $\alpha_n \to 0$ and
\[
W^M_{\epsilon_n, \eta_n}(\tau, \phi) \leq \Gamma(\tau, \phi) + \alpha_n \text{ for all } (\tau, \phi) \in [0, T] \times (\mathbf{C} \oplus \mathbf{B}).
\]
We know that
\[ S_M(\varepsilon_n, \eta_n, \tau_n, \phi_n, W^M_{\varepsilon_n, \eta_n}(\tau_n, \phi_n)) = 0. \]

The monotonicity of \( S_M \) and (27) implies
\[
S_M(\varepsilon_n, \eta_n, \tau_n, \phi_n, \Gamma(\tau_n, \phi_n) + \alpha_n, \Gamma + \alpha_n) \\
\leq S_M(\varepsilon_n, \eta_n, \tau_n, \phi_n, W^M_{\varepsilon_n, \eta_n}(\tau_n, \phi_n)) = 0. \tag{28}
\]

Therefore,
\[
\lim_{n \to \infty} S_M(\varepsilon_n, \eta_n, \tau_n, \phi_n, \Gamma(\tau_n, \phi_n) + \alpha_n, \Gamma + \alpha_n) \leq 0,
\]

Therefore using Lemma 3.3 we obtain,
\[
\min \left\{ \Gamma^*_M(t, \psi) - (\Psi(t, \psi) \wedge M), \right. \\
\rho \Gamma^*_M(t, \psi) - \frac{\partial \Gamma}{\partial t}(t, \psi) - S(\Gamma)(t, \psi) - D\Gamma(t, \psi)(f(t, \psi)1_{(0)}) \\
- \frac{1}{2} \sum_{i=1}^m D^2 \Gamma(t, \psi)(g(t, \psi) e_i 1_{(0)}, g(t, \psi) e_i 1_{(0)}) - (L(t, \psi) \wedge M) \left\} \right. \\
= \lim_{n \to \infty} S_M(\varepsilon_n, \eta_n, \tau_n, \phi_n, \Gamma(\tau_n, \phi_n) + \alpha_n, \Gamma + \alpha_n) \leq 0. \tag{29}
\]

This proves that \( W^*_M \) is a viscosity subsolution of (10).

Similarly we can prove that \( W_{sM} \) is a viscosity supersolution. By virtue of the Comparison Principle (Theorem 4.7 in [4]), we can get that
\[
W_{sM}(t, \psi) \geq W^*_M(t, \psi), \quad \forall (t, \psi) \in [0, T] \times C. \tag{30}
\]

On the other hand, by the definitions of \( W_{sM} \), \( W^*_M \), it is easy to see that
\[
W_{sM}(t, \psi) \leq W^*_M(t, \psi), \quad \forall (t, \psi) \in [0, T] \times C.
\]

Combined with (30), the above implies
\[
W_{sM}(t, \psi) = W^*_M(t, \psi), \quad \forall (t, \psi) \in [0, T] \times C.
\]

Since \( W_{sM} \) is a viscosity supersolution and \( W^*_M \) is a viscosity subsolution, they are also viscosity solutions of (10). Now, using the uniqueness of the viscosity solution (10), we see that \( V_M = W^*_M = W_{sM} \). Therefore, we conclude that the sequence \( (W^*_M)_{\varepsilon, \eta} \) converges locally uniformly to \( V_M \) as desired. \( \square \)
4 The Computational Algorithm

Based on the results obtained in the last section, we can construct the computational algorithm to obtain a numerical solution. For example, one algorithm can be like the following:

Step 0. Choose any function $W(0) \in \mathcal{C}([0, T] \times C \oplus B)_b$;

Step 1. Pick the starting values for $\epsilon(1), \eta(1)$. For example, we can choose $\epsilon(1) = 10^{-2}, \eta(1) = 10^{-3}$;

Step 2. For the given $\epsilon, \eta > 0$, compute the function $W^{(1)}_{\epsilon(1), \eta(1)} \in \mathcal{C}([0, T] \times C \oplus B)_b$ by the following formula

$$W^{(1)}_{\epsilon(1), \eta(1)} = T_{\epsilon(1), \eta(1)}W(0),$$

where $T_{\epsilon(1), \eta(1)}$, which is defined on $C_b([0, T] \times C \oplus B)$, is given by (21);

Step 3. Repeat Step 2 for $i = 2, 3, \cdots$ using

$$W^{(i)}_{\epsilon(1), \eta(1)} = T_{\epsilon(1), \eta(1)}W^{(i-1)}_{\epsilon(1), \eta(1)},$$

where $T_{\epsilon(1), \eta(1)}$, which is defined on $C_b([0, T] \times C \oplus B)$, is given by (21). Stop the iteration when

$$\|W^{i+1}_{\epsilon(1), \eta(1)}(t, \psi) - W^i_{\epsilon(1), \eta(1)}(t, \psi)\| \leq \delta_1,$$

where $\delta_1$ is a preselected number which is small enough to achieve the accuracy we want. Denote the final solution by $W_{\epsilon(1), \eta(1)}(t, \psi)$.

Step 4. Choose two sequences of $\epsilon(k)$ and $\eta(k)$, such that

$$\lim_{k \to \infty} \epsilon(k) = \lim_{k \to \infty} \eta(k) = 0.$$  

For example, we may choose $\epsilon(k) = \eta(k) = 10^{-(2+k)}$. Now repeat Step 2 and Step 3 for each $\epsilon(k), \eta(k)$ until

$$\|W_{\epsilon(k+1), \eta(k+1)}(t, \psi) - W_{\epsilon(k), \eta(k)}(t, \psi)\| \leq \delta_2,$$

where $\delta_2$ is chosen to obtain the expected accuracy.

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References


