Optimal Control of Stochastic Functional Differential Equations with a Bounded Memory

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Abstract

This paper treats a finite time horizon optimal control problem in which the controlled state dynamics is governed by a general system of stochastic functional differential equations with a bounded memory. An infinite-dimensional HJB equation is derived using a Bellman-type dynamic programming principle. It is shown that the value function is the unique viscosity solution of the HJB equation.

Keywords: Stochastic control, stochastic functional differential equations, viscosity solutions.
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1 Introduction

Optimal control of Itô’s diffusion processes have been extensively studied in the literature, see Fleming and Rishel [4] for the classical theory, Fleming and Soner [5] and the references contained herein for the viscosity solution approach.

However, in many real world applications (see Kolmanovskii and Shaikhet [9]), these physical systems can only be modelled by stochastic dynamical systems whose evolution depend on the past history of the state. Such models are referred to as stochastic (retarded) functional differential equations (see Mohammed [16], [17] for an introduction of these models). The linear-quadratic regulatory problem involving stochastic delay equations was first studied in Kolmanovskii and Maizenberg [8], and optimal control problems for a class of nonlinear stochastic equations that involve a continuous delay of the following type

\[
dX(s) = \alpha(s, X(s), Y(s), u(s)) ds \\
+ \beta(s, X(s), Y(s), u(s)) dW(s), \quad s \in [t, T],
\]

have been studied in recent literature (see e.g. Elsanousi [2], Elsanousi et al [3], and Larssen [10], Oksendal and Sulem [19]), in which \( Y(s) = \int_{-r}^{0} e^{-\delta \theta} X(s + \theta)d\theta \).

The purpose of this paper is to investigate a finite time horizon optimal control problem for a general system of stochastic functional differential equations that include (1) as a special case. We use the viscosity solution concept introduced by Crandall and Lions [1], [14], [15] in order to characterize the value function as the unique viscosity solution of the associated HJB equation.

This paper is organized as follows. Notation and the statement of the problem are contained in Section 2. In Section 3, the infinite dimensional Hamilton-Jacobi-Bellman (HJB) equation for the value function is heuristically derived using Bellman’s type dynamic programming principle first obtained in Larssen [10]. In Section 4, the continuity of the value function is proved. However, the value function is not known to be smooth and therefore it may not be a classical solution of the HJB in general cases. It is shown in Section 4 that the value function is a viscosity solution of the HJB equation. The uniqueness result for viscosity solution of the HJB equation is given in Section 5.
2 Problem Formulation

Let $T > 0$ denote a fixed terminal time, and let $t ∈ [0, T]$ denote an initial time. We study the finite time horizon optimal control problem for a general system of stochastic functional differential equations on the interval $[t, T]$. Let $r > 0$ be a fixed constant, and let $J = [−r, 0]$ denote the duration of the bounded memory of the equations considered in this paper. For the sake of simplicity, denote $C(J; ℜ^n)$, the space of continuous functions $φ : J → ℜ^n$, by $C$. Note that $C$ is a real separable Banach space under the sup-norm defined by

$$
\|φ\| = \sup_{t \in J} |φ(t)|, \quad φ ∈ C,
$$

where $| · |$ is the Euclidean norm in $ℜ^n$.

We denote by $(· | ·)$ the inner product in $L^2(J; ℜ^n)$, and $⟨·, ·⟩$ the inner product in $ℜ^n$. Given $φ$ and $ψ$ in $C$, we have defined as follows,

$$
(φ | ψ) = \int_{−r}^{0} ⟨φ(s), ψ(s)⟩ ds, \quad \text{and} \quad \|φ\|_2 = (φ | φ)^{\frac{1}{2}}.
$$

Note that the space $C$ can be continuously embedded into $L^2(J; ℜ^n)$.

**Convention 2.1** Throughout the end, we use the following conventional notation for functional differential equations (see Hale [6]):

If $ψ ∈ C([-r, ∞); ℜ^n)$ and $t ∈ ℜ_+$, let $ψ_t ∈ C$ be defined by $ψ_t(θ) = ψ(t + θ)$, $θ ∈ J$.

Throughout the end, let $\{W(t), t ≥ 0\}$ be a certain $m$-dimensional standard Brownian motion defined on a complete filtered probability space $(Ω, ℱ, P, ℋ)$, where $ℋ = \{ℱ(t), t ≥ 0\}$ is the $P$-augmentation of the natural filtration $\{ℱ^W(t), t ≥ 0\}$ generated by the Brownian motion $\{W(t), t ≥ 0\}$, i.e., if $t ≥ 0$,

$$
ℱ^W(t) = σ\{W(s), 0 ≤ s ≤ t\}
$$

and

$$
ℱ(t) = ℱ^W(t) ∨ \{A ⊂ Ω|∃B ∈ ℱ \text{ such that } A ⊂ B \text{ and } P(B) = 0\}
$$

where the operator $∨$ denotes that $ℱ(t)$ is the smallest $σ$-algebra such that $ℱ^W(t) ⊂ ℱ(t)$ and

$$
\{A ⊂ Ω|∃B ∈ ℱ \text{ such that } A ⊂ B \text{ and } P(B) = 0\} ⊂ ℱ(t).
$$
Let $L^2(\Omega, \mathbb{C})$ be the space of $\mathbb{C}$-valued random variables $\Xi : \Omega \to \mathbb{C}$ such that

$$\|\Xi\|_{L^2} = \left\{ \int_\Omega \|\Xi(\omega)\|^2 dP(\omega) \right\}^{1/2} < \infty.$$ 

In addition, let $L^2(\Omega, \mathbb{C}; \mathcal{F}(t))$ be those $\Xi \in L^2(\Omega, \mathbb{C})$ which are $\mathcal{F}(t)$-measurable.

We consider the following system of controlled stochastic functional differential equations with a bounded memory:

$$dX(s) = f(s, X_s, u(s))ds + g(s, X_s, u(s))dW(s), \quad s \in [t, T], \quad (2)$$

with the initial function $X_t = \psi_t$, where $\psi_t \in L^2(\Omega, \mathbb{C}; \mathcal{F}(t))$ and $u(\cdot) = \{u(s), s \in [t, T]\}$ is a control process taking values in a compact set $U$ (of an Euclidean space). The functions, $f : [0, T] \times \mathbb{C} \times U \to \mathbb{R}^n$ and $g : [0, T] \times \mathbb{C} \times U \to \mathbb{R}^n \times m$ are given deterministic functions.

**Definition 2.2** Given the $m$-dimensional standard Brownian motion $\{W(s), s \in [0, T]\}$ and the control process $\{u(s), s \in [t, T]\}$, a process $\{X(s; t, \psi_t, u(\cdot)), s \in [t - r, T]\}$ is said to be a (strong) solution of the controlled equation (2) on the interval $[t - r, T]$ and through the initial datum $(t, \psi_t) \in [0, T] \times L^2(\Omega, \mathbb{C}; \mathcal{F}(t))$ if it satisfies the following conditions:

1. $X_t(\cdot; t, \psi_t, u(\cdot)) = \psi_t$;
2. $X(s; t, \psi_t, u(\cdot))$ is $\mathcal{F}(s)$-measurable for each $s \in [t, T]$;
3. The process $\{X(s; t, \psi_t, u(\cdot)), s \in [t, T]\}$ is continuous and satisfies the following stochastic integral equation P-a.s.

$$X(s) = \psi_t(0) + \int_t^s f(\lambda, X\lambda, u(\lambda))d\lambda + \int_t^s g(\lambda, X\lambda, u(\lambda))dW(\lambda). \quad (3)$$

In addition, the solution process $\{X(s; t, \psi_t, u(\cdot)), s \in [t - r, T]\}$ is said to be (strongly) unique if $\{\tilde{X}(s; t, \psi_t, u(\cdot)), s \in [t - r, T]\}$ is also a solution of (2) on $[t - r, T]$ with the control process $u(\cdot)$ and through the same initial datum $(t, \psi_t)$, then

$$P\{X(s; t, \psi_t, u(\cdot)) = \tilde{X}(s; t, \psi_t, u(\cdot)), \forall s \in [t, T]\} = 1.$$ 

**Definition 2.3** For each $t \in [0, T]$, a 5-tuple $\alpha = (\Omega, \mathcal{F}, P, W(\cdot), u(\cdot))$ is said to be an admissible control if it satisfies the following conditions:
1. \((\Omega, \mathcal{F}, P)\) is a complete probability space.

2. \(W(\cdot) = \{W(s), s \in [0, T]\}\) is an \(m\)-dimensional standard Brownian motion on \((\Omega, \mathcal{F}, P)\) over \([t, T]\) with \(W(t) = 0\) a.s., and \(\mathcal{F}(t, s) = \sigma\{W(\tau), t \leq \tau \leq s\}\) augmented by the \(P\)-null sets in \(\mathcal{F}\).

3. \(u : [t, T] \times \Omega \to U\) is an \(\{\mathcal{F}(t, s), s \in [t, T]\}\)-adapted process on \((\Omega, \mathcal{F}, P)\) that is right-continuous at the initial time \(t\).

4. Under the control process \(u(\cdot) = \{u(s), s \in [t, T]\}\), equation (2) admits a unique strong solution \(X_{t, \psi, u(\cdot)}(\cdot) = \{X(s; t, \psi, u(\cdot)), s \in [t, T]\}\) on \((\Omega, \mathcal{F}, P; \{\mathcal{F}(t, s), s \in [t, T]\})\) through each initial datum \((t, \psi) \in [0, T] \times C\).

5. The control process \(u(\cdot)\) is such that

\[
\mathbb{E}\left[ \int_t^T |L(s, X_s(t, \psi, u(\cdot)), u(s))| ds + |\Psi(X_T(t, \psi, u(\cdot)))| \right] < \infty,
\]

where \(L : [0, T] \times C \times U \to \mathbb{R}\) and \(\Psi : C \to \mathbb{R}\) represent the running and terminal cost functions, respectively.

The collection of admissible controls \(\alpha = (\Omega, \mathcal{F}, P, W(\cdot), u(\cdot))\) over the interval \([t, T]\) shall be denoted by \(U[t, T]\).

We shall write \(u(\cdot) \in U[t, T]\) or \(\alpha = (\Omega, \mathcal{F}, P, W(\cdot), u(\cdot)) \in U[t, T]\) interchangeably, whenever there is no danger of ambiguity.

Throughout the end, we assume that the functions \(f : [0, T] \times C \times U \to \mathbb{R}^n\), and \(g : [0, T] \times C \times U \to \mathbb{R}^{n \times m}\) satisfy the following conditions. (See Mohammed [16, 17].) The functions \(f\) and \(g\) are continuous and they satisfy the following linear growth and Lipschitz conditions.

**Assumption 2.4** There exists a constant \(\Lambda > 0\) such that

\[
|f(t, \varphi, u) - f(t, \phi, u)| + |g(t, \varphi, u) - g(t, \phi, u)| \leq \Lambda \|\varphi - \phi\|, \\
\forall (t, \varphi, u), (t, \phi, u) \in [0, T] \times C \times U.
\]

**Assumption 2.5** There exists a constant \(K > 0\) such that

\[
|f(t, \phi, u)| + |g(t, \phi, u)| \leq K(1 + \|\phi\|), \forall (t, \phi, u) \in [0, T] \times C \times U.
\]
Given an admissible control $u(\cdot) \in U[t, T]$, let $X^{t, \psi, u(\cdot)}(\cdot) = \{X(s; t, \psi, u(\cdot)), s \in [t, T]\}$ be the solution of (2) through the initial datum $(t, \psi) \in [0, T] \times \mathbb{C}$. We again consider the corresponding $\mathbb{C}$-valued process $\{X_s(t, \psi, u(\cdot)), s \in [t, T]\}$ defined by

$$X_s(\theta; t, \psi, u(\cdot)) = X(s + \theta; t, \psi, u(\cdot)), \quad \theta \in \mathbb{J}. \quad (4)$$

For notational simplicity, we often write $X(s) = X(s; t, \psi, u(\cdot))$ and $X_s = X_s(t, \psi, u(\cdot))$ for $s \in [t, T]$ whenever there is no danger of ambiguity.

It can be shown under Assumptions 2.4-2.5 that the $\mathbb{C}$-valued process $\{X_s(t, \psi, u(\cdot)), s \in [t, T]\}$ is a Markov process (see Mohammed [16], [17]).

Let $L$ and $\Psi$ be two continuous real-valued functions on $[0, T] \times \mathbb{C} \times U$ and $[0, T] \times \mathbb{C}$, respectively. Moreover, we assume that they both have at most polynomial growth in $L^2(\mathbb{J}; \mathbb{R})$. In other words, there exist constants $\Lambda, k$ such that

$$|L(t, \phi, u)| \leq \Lambda(1 + \|\phi\|_2)^k, \quad \text{and} \quad |\Psi(t, \phi)| \leq \Lambda(1 + \|\phi\|_2)^k,$$

for all $(t, \phi, u) \in [0, T] \times \mathbb{C} \times U$, for some positive integer $k$.

Given any initial data $(t, \psi) \in [0, T] \times \mathbb{C}$ and any admissible control $u(\cdot) \in U[t, T]$, we define the objective function

$$J(t, \psi; u(\cdot)) \equiv \mathbb{E}\left[ \int_t^T e^{-\rho(s-t)} L(s, X_s(t, \psi, u(\cdot)), u(s))ds + e^{-\rho(T-t)} \Psi(X_T(t, \psi, u(\cdot))) \right], \quad (5)$$

where $\rho > 0$ denotes a discount factor. For each initial datum $(t, \psi) \in [0, T] \times \mathbb{C}$, the optimal control problem is to find $u(\cdot) \in U[t, T]$ so as to maximize the objective function $J$. In this case, the value function $V : [0, T] \times \mathbb{C} \rightarrow \mathbb{R}$ is defined to be

$$V(t, \psi) = \sup_{u(\cdot) \in U[t, T]} J(t, \psi; u(\cdot)). \quad (6)$$

3 The HJB Equation

3.1 The Infinitesimal Generator

Let $\mathbb{C}^*$ and $\mathbb{C}^\dagger$ be the space of bounded linear functionals $\Phi : \mathbb{C} \rightarrow \mathbb{R}$ and bounded bilinear functionals $\tilde{\Phi} : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{R}$, of the space $\mathbb{C}$, respectively.
They are equipped with the operator norms which will be, respectively, denoted by $\| \cdot \|_*$ and $\| \cdot \|^\dagger$.

Let $B = \{ v1_{(0)} | v \in \mathbb{R}^n \}$, where $1_{(0)} : [-r,0] \to \mathbb{R}$ is defined by

$$1_{(0)}(\theta) = \begin{cases} 0 & \text{for } \theta \in [-r,0), \\ 1 & \text{for } \theta = 0. \end{cases}$$

We form the direct sum

$$C \oplus B = \{ \phi + v1_{(0)} | \phi \in C, \ v \in \mathbb{R}^n \}$$

and equip it with the norm $\| \cdot \|$ defined by

$$\| \phi + v1_{(0)} \| = \sup_{\theta \in [-r,0]} |\phi(\theta)| + |v|, \ \phi \in C, \ v \in \mathbb{R}^n.$$

Note that for each sufficiently smooth function $\Phi : C \to \mathbb{R}$, its first order Fréchet derivative (with respect to $\phi \in C$), $D\Phi(\varphi) \in C^*$, has a unique and continuous linear extension $D\Phi(\varphi) \in (C \oplus B)^*$. Similarly, its second order Fréchet derivative, $D^2\Phi(\varphi) \in C^\dagger$, has a unique and continuous linear extension $D^2\Phi(\varphi) \in (C \oplus B)^\dagger$. In above, $(C \oplus B)^*$ and $(C \oplus B)^\dagger$ are spaces of bounded linear and bilinear functionals of $C \oplus B$, respectively. (See Lemma (3.1) and Lemma (3.2) on pp 79-83 of Mohammed [16] for details).

For a Borel measurable function $\Phi : C \to \mathbb{R}$, we also define

$$S(\Phi)(\phi) = \lim_{h \to 0^+} \frac{1}{h} \left[ \Phi(\tilde{\phi}_h) - \Phi(\phi) \right]$$

for all $\phi \in C$, where $\tilde{\phi} : [-r,T] \to \mathbb{R}^n$ is an extension of $\phi$ defined by

$$\tilde{\phi}(t) = \begin{cases} \phi(t) & \text{if } t \in [-r,0) \\ \phi(0) & \text{if } t \geq 0, \end{cases}$$

and again $\tilde{\phi}_t \in C$ is defined by

$$\tilde{\phi}_t(\theta) = \tilde{\phi}(t + \theta), \ \theta \in [-r,0].$$

Let $D(S)$, the domain of the operator $S$, be the set of $\Phi : C \to \mathbb{R}$ such that the above limit exists for each $\phi \in C$.

Throughout the end, let $C^{1,2}_{lip}(0,T] \times C)$ be the space of functions $\Phi : [0,T] \times C \to \mathbb{R}$ such that $\frac{\partial \Phi}{\partial t} : [0,T] \times C \to \mathbb{R}$ and $D^2\Phi : [0,T] \times C \to C^\dagger$ exist and are continuous and satisfy the following Lipschitz condition:

$$\| D^2\Phi(t,\phi) - D^2\Phi(t,\varphi) \|^\dagger \leq K \| \phi - \varphi \| \ \forall t \in [0,T], \ \phi, \varphi \in C.$$
Theorem 3.1 Suppose that $\Phi \in C_{lip}^{1,2}([0,T] \times \mathbb{C}) \cap \mathcal{D}(S)$. Let $u(\cdot) \in \mathcal{U}[t,T]$ and $\{X_s, s \in [t,T]\}$ be the $\mathbb{C}$-valued Markov solution process of equation (2) with the initial data $(t,\varphi_t) \in [0,T] \times \mathbb{C}$.

Then
\[
\lim_{\epsilon \downarrow 0} E[\Phi(t + \epsilon, X_{t+\epsilon}) - \Phi(t, \varphi_t)] = \partial_t \Phi(t, \varphi_t) + S(\Phi)(t, \varphi_t) + D\Phi(t, \varphi_t)(f(t, \varphi_t, u(t))1_{\{0\}})
\]
\[
+ \frac{1}{2} \sum_{j=1}^{m} D^2 \Phi(t, \varphi_t)(g(t, \varphi_t, u(t))(e_j)1_{\{0\}}, g(t, \varphi_t, u(t))1_{\{0\}}),
\]
ge where $e_j, j = 1, 2, \cdots, m$, is the $j$th unit vector of the standard basis in $\mathbb{R}^m$.

Proof. One can refer to Mohammed [16], [17].

3.2 Heuristic Derivation of the HJB Equation

In this subsection, we re-state the dynamic programming principle described in Larssen [10].

Theorem (Larssen). Let Assumptions 2.4-2.5 hold. Then for any $(t,\psi) \in [0,T] \times \mathbb{C}$ and $F(t)$-stopping time $\tau \in [t,T]$,

\[
V(t,\psi) = \sup_{u(\cdot) \in \mathcal{U}[t,T]} E \left[ \int_t^\tau e^{-\rho(s-t)} L(s, X_s(t,\psi, u(\cdot)), u(s)) ds + e^{-\rho(\tau-t)} V(\tau, X_\tau(t,\psi, u(\cdot))) \right].
\]

Let $v \in \mathcal{U}$. We define:

\[
\mathcal{A}^v V(t,\psi) \equiv S(V)(t,\psi) + DV(t,\psi)(f(t,\psi,v)1_{\{0\}})
\]
\[
+ \frac{1}{2} \sum_{i=1}^{m} D^2 V(t,\psi)(g(t,\psi,v)e_i1_{\{0\}},g(t,\psi,v)e_i1_{\{0\}}).
\]

We assume that for every $v \in \mathcal{U}$, the domain of the generator $\mathcal{A}^v$ is large enough to contain $C^{1,2}_{lip}([0,T] \times \mathbb{C}) \cap \mathcal{D}(S)$.

From the dynamic programming principle (Theorem (Larssen)), if we take a constant control $u(\cdot) \equiv v$, then $\forall \delta \geq 0$

\[
V(t,\psi) \geq E \left[ \int_t^{t+\delta} e^{-\rho(s-t)} L(s, X_s(t,\psi, v), v) ds + e^{-\rho \delta} V(t+\delta, X_{t+\delta}(t,\psi, v)) \right].
\]
From this principle, we have

\[
0 \geq \lim_{\delta \to 0} \frac{1}{\delta} \mathbb{E} \left[ \int_{t}^{t+\delta} e^{-\rho(s-t)} L(s, X_s(t, \psi, v), v) ds 
+ e^{-\rho\delta} V(t + \delta, X_{t+\delta}(t, \psi, v)) - V(t, \psi) \right]
= -\rho V(t, \psi) + \frac{\partial V}{\partial t}(t, \psi) + A^v V(t, \psi) + L(t, \psi, v)
\]

for all \((t, \psi) \in [0, T] \times \mathbb{C}\), provided that \(V \in C^{1,2}_{lip}([0, T] \times \mathbb{C}) \cap \mathcal{D}(S)\).

Moreover, if \(u^*(\cdot) \in \mathcal{U}[t, T]\) is the optimal control policy which satisfies \(\lim_{s \to t} u^*(s) = v^*\), we should have that \(\forall \delta \geq 0\)

\[
V(t, \psi) = \mathbb{E} \left[ \int_{t}^{t+\delta} e^{-\rho(s-t)} L(s, X^*_s(t, \psi, u^*(s)), v^*) ds + e^{-\rho\delta} V(t + \delta, X^*_s(t, \psi, u^*)) \right],
\]

where \(X^*_s = X^*_s(t, \psi, u^*(\cdot))\) is the \(\mathbb{C}\)-valued solution process corresponding to the initial datum \((t, \psi)\) and the optimal control \(u^*(\cdot) \in \mathcal{U}[t, T]\). Similarly, under strong assumption on \(u^*(\cdot)\) (including the right-continuity at the initial time \(t\)), we can get

\[
0 = -\rho V(t, \psi) + \frac{\partial V}{\partial t}(t, \psi) + A^v V(t, \psi) + L(t, \psi, v^*).
\]

The inequality (9) and (11) are equivalent to the HJB equation

\[
0 = -\rho V(t, \psi) + \frac{\partial V}{\partial t}(t, \psi) + \max_{v \in \mathcal{U}} [A^v V(t, \psi) + L(t, \psi, v)].
\]

We therefore have the following result.

**Theorem 3.2** Suppose \(V\) is the value function defined by (6) and satisfies \(V \in C^{1,2}_{lip}([0, T] \times \mathbb{C}) \cap \mathcal{D}(S)\). Then the value function \(V\) satisfies the following HJB equation:

\[
\rho V(t, \psi) - \frac{\partial V}{\partial t}(t, \psi) - \max_{v \in \mathcal{U}} [A^v V(t, \psi) + L(t, \psi, v)] = 0
\]

on \([0, T] \times \mathbb{C}\), and \(V(T, \psi) = \Psi(\psi), \ \forall \psi \in \mathbb{C}\).

Note that it is not known that the value function \(V\) satisfies the necessary smoothness condition \(V \in C^{1,2}_{lip}([0, T] \times \mathbb{C}) \cap \mathcal{D}(S)\). Therefore, in general we need to consider viscosity solution instead of a classical solution for HJB equation (13). In fact, it will be shown that the value function is a unique viscosity solution of the HJB equation (13). These results shall be given in the next two sections.
4 Viscosity Solution of the HJB Equation

In this section, we shall show that the value function $V$ defined by (6) is actually a viscosity solution of the HJB equation (13). First, let us define the viscosity solution of (13) as follows.

**Definition 4.1** Let $w \in C([0,T] \times \mathbb{C})$. We say that $w$ is a viscosity subsolution of (13) if, for every $\Gamma \in C^{1,2}_{\text{lip}}([0,T] \times \mathbb{C}) \cap D(S)$, for $(t,\psi) \in [0,T] \times \mathbb{C}$ satisfying $\Gamma \geq w$ on $[0,T] \times \mathbb{C}$ and $\Gamma(t,\psi) = w(t,\psi)$, we have

$$\rho \Gamma(t,\psi) - \frac{\partial \Gamma}{\partial t}(t,\psi) - \max_{v \in U} [A^v \Gamma(t,\psi) + L(t,\psi,v)] \leq 0.$$  

We say that $w$ is a viscosity super solution of (13) if, for every $\Gamma \in C^{1,2}_{\text{lip}}([0,T] \times \mathbb{C}) \cap D(S)$, and for $(t,\psi) \in [0,T] \times \mathbb{C}$ satisfying $\Gamma \leq w$ on $[0,T] \times \mathbb{C}$ and $\Gamma(t,\psi) = w(t,\psi)$, we have

$$\rho \Gamma(t,\psi) - \frac{\partial \Gamma}{\partial t}(t,\psi) - \max_{v \in U} [A^v \Gamma(t,\psi) + L(t,\psi,v)] \geq 0.$$  

We say that $w$ is a viscosity solution of (13) if it is both a viscosity supersolution and a viscosity subsolution of (13).

For our value function $V$ defined by (6), we now show that it has the following property.

**Lemma 4.2** The value function $V$ defined in (6) is continuous and there exists a constant $\Lambda > 0$ and a positive integer $k$ such that, for every $(t,\phi) \in [0,T] \times \mathbb{C}$,

$$|V(t,\phi)| \leq \Lambda(1 + \|\phi\|_2)^k. \quad (14)$$

and there exists a constant $K > 0$ such that

$$|V(s,\phi) - V(s,\varphi)| \leq K\|\phi - \varphi\|, \quad \forall (s,\phi),(s,\varphi) \in [0,T] \times \mathbb{C}. \quad (15)$$

**Proof.** It is clear that $V$ has at most a polynomial growth, since $L$ and $\Phi$ have at most a polynomial growth. The fact that the value function $V$ satisfies (15) follows from Lemma 5.1 (Paper 2, pp. 11) of Larssen [10].

We next show the continuity of $V(t,\psi)$ with respect to $t$. Let $\Xi_1(s) = X_s(t_1,\psi,u(\cdot))$ and $\Xi_2(s) = X_s(t_2,\psi,u(\cdot))$, $s \in [t,T]$, be two $\mathbb{C}$-valued solutions of (2) with $u(\cdot) \in \mathcal{U}([0,T]$ and the initial data $(t_1,\psi)$ and $(t_2,\psi)$,
respectively.

Without loss of generality let us assume that \(0 \leq t_1 < t_2 \leq T\) then

\[
J(t_1, \psi; u(\cdot)) - J(t_2, \psi; u(\cdot)) = E \left[ \int_{t_1}^{t_2} e^{-\rho(\xi-t_1)} [L(\xi, \Xi_1(\xi), u(\xi))] d\xi + \int_{t_2}^{T} e^{-\rho(\xi-t_2)} [L(\xi, \Xi_1(\xi), u(\xi)) - L(\xi, \Xi_2(\xi), u(\xi))] d\xi + e^{-\rho(T-t_1)} \Psi(\Xi_1(T)) - e^{-\rho(T-t_2)} \Psi(\Xi_2(T)) \right]
\]

Therefore, there exists a constant \(\Lambda > 0\) such that

\[
|J(t_1, \psi, u(\cdot)) - J(t_2, \psi, u(\cdot))| \leq \Lambda \left( |t_1 - t_2| E\|\Xi_1(T)\| + E\|\Xi_1(T) - \Xi_2(T)\| \right)
\]

Now let \(\varepsilon > 0\) be a small enough constant. Using the compactness of \([0, T]\) and the uniform continuity of the trajectory map in \(t\), we know that there exists \(\eta > 0\) such that if \(|t_1 - t_2| < \eta\) then \(E\|\Xi_1(s) - \Xi_2(s)\| \leq \frac{\varepsilon}{2\Lambda} \) for all \(s \in [t_2, T]\). Therefore, for

\[
|t_1 - t_2| < \min(\eta, \frac{\varepsilon}{2\Lambda E\|\Xi_1(T)\|})
\]

we have

\[
|J(t_1, \psi, u(\cdot)) - J(t_2, \psi, u(\cdot))| \leq \varepsilon + \frac{\varepsilon}{2} = \varepsilon.
\]

Consequently,

\[
|V(t_1, \psi) - V(t_2, \psi)| \leq \varepsilon.
\]

\[\square\]

**Theorem 4.3** The value function \(V\) is a viscosity solution of the HJB equation

\[
\rho V(t, \psi) - \frac{\partial V}{\partial t}(t, \psi) - \max_{v \in U} [A^v V(t, \psi) + L(t, \psi, v)] = 0
\]

on \([0, T] \times C\), and \(V(T, \psi) = \Psi(\psi), \ \forall \psi \in C\).
Proof. Let $\Gamma \in C^{1,2}_{lip}([0, T] \times \mathbb{C}) \cap \mathcal{D}(\mathcal{S})$. For $(t, \psi) \in [0, T] \times \mathbb{C}$ such that $\Gamma \leq V$ on $[0, T] \times \mathbb{C}$ and $\Gamma(t, \psi) = V(t, \psi)$, we want to prove the viscosity supersolution inequality, i.e.,

$$
\rho \Gamma(t, \psi) - \frac{\partial \Gamma}{\partial t}(t, \psi) - \max_{v \in U} [A^u \Gamma(t, \psi) + L(t, \psi, v)] \geq 0.
$$

(19)

Let $u(\cdot) \in U[t, T]$. Since $\Gamma \in C^{1,2}_{lip}([0, T] \times \mathbb{C}) \cap \mathcal{D}(\mathcal{S})$, (by virtue of Theorem 3.1 pp. 78 of Mohammed [16]) for $t \leq s \leq T$, we have

$$
E\left[e^{-\rho(s-t)}\Gamma(s, X_s(t, \psi, u(\cdot))) - \Gamma(t, \psi)\right]
= E\left[\int_t^s e^{-\rho(\xi-t)} \left(\frac{\partial \Gamma}{\partial \xi}(\xi, X_\xi(t, \psi, u(\cdot))) + A^u \Gamma(\xi, X_\xi(t, \psi, u(\cdot))) - \rho \Gamma(\xi, X_\xi(t, \psi, u(\cdot)))\right)d\xi\right].
$$

(20)

On the other hand, for any $s \in [t, T]$, the dynamic programming principle (Theorem (Larssen)) gives,

$$
V(t, \psi) = \max_{u(\cdot) \in U[t, T]} E\left\{ \int_t^s e^{-\rho(\xi-t)} L(\xi, X_\xi(t, \psi, u(\cdot)), u(\xi))d\xi + e^{-\rho(s-t)} V(s, X_s(t, \psi, u(\cdot))) \right\}.
$$

(21)

Therefore, we have

$$
V(t, \psi) \geq E\left[\int_t^s e^{-\rho(\xi-t)} L(\xi, X_\xi(t, \psi, u(\cdot)), u(\xi))d\xi\right] + E\left[e^{-\rho(s-t)} V(s, X_s(t, \psi, u(\cdot)))\right].
$$

(22)

By virtue of (20) and using $\Gamma \leq V, \Gamma(t, \psi) = V(t, \psi)$, we can get

$$
0 \geq E\left[\int_t^s e^{-\rho(\xi-t)} L(\xi, X_\xi(t, \psi, u(\cdot)), u(\xi))d\xi\right] + E\left[e^{-\rho(s-t)} V(s, X_s(t, \psi, u(\cdot)))\right] - V(t, \psi)
\geq E\left[\int_t^s e^{-\rho(\xi-t)} L(\xi, X_\xi(t, \psi, u(\cdot)), u(\xi))d\xi\right] + E\left[e^{-\rho(s-t)} \Gamma(s, X_s(t, \psi, u(\cdot)))\right] - \Gamma(t, \psi)
\geq E\left[\int_t^s e^{-\rho(\xi-t)} \left[- \rho \Gamma(\xi, X_\xi(t, \psi, u(\cdot))) + \frac{\partial \Gamma(\xi, X_\xi(t, \psi, u(\cdot)))}{\partial \xi} + A^u \Gamma(\xi, X_\xi(t, \psi, u(\cdot))) + L(\xi, X_\xi(t, \psi, u(\cdot)), u(\xi))\right]d\xi\right].
$$
Dividing both sides by \((s - t)\), we have

\[
0 \leq \mathbb{E} \left[ \frac{1}{s - t} \int_t^s e^{-\rho(s - t)} \left( \rho \Gamma(\xi, X_\xi(t, \psi, u(\cdot))) - \frac{\partial \Gamma(\xi, X_\xi(t, \psi, u(\cdot)))}{\partial \xi} - A^u(\xi) \Gamma(\xi, X_\xi(t, \psi, u(\cdot))) - L(\xi, X_\xi(t, \psi, u(\cdot)), u(\xi)) \right) d\xi \right].
\] (23)

Now let \(s \downarrow t\) in (23) and \(\lim_{s \downarrow t} u(s) = v\), and we obtain

\[
\rho \Gamma(t, \psi) - \frac{\partial \Gamma}{\partial t}(t, \psi) - [A^v \Gamma(t, \psi) + L(t, \psi, v)] \geq 0.
\] (24)

Since \(v \in U\) is arbitrary, we deduce the inequality (19).

Next we want to prove that \(V\) is a viscosity subsolution. Let \(\Gamma \in C^{1,2}_{lip}([0, T] \times \mathbb{C}) \cap D(S)\). For \((t, \psi) \in [0, T] \times \mathbb{C}\) satisfying \(\Gamma \geq V\) on \([0, T] \times \mathbb{C}\) and \(\Gamma(t, \psi) = V(t, \psi)\), we want to prove that

\[
\rho \Gamma(t, \psi) - \frac{\partial \Gamma}{\partial t}(t, \psi) - \max_{v \in U} [A^v \Gamma(t, \psi) + L(t, \psi, v)] \leq 0.
\] (25)

We assume the contrary, and try to obtain a contradiction. Let suppose that, there exit \((t, \psi) \in [0, T] \times \mathbb{C}\), \(\Gamma \in C^{1,2}_{lip}([0, T] \times \mathbb{C}) \cap D(S)\), with \(\Gamma \geq V\) on \([0, T] \times \mathbb{C}\) and \(\Gamma(t, \psi) = V(t, \psi)\), and \(\delta > 0\) such that for all control \(u(\cdot) \in U[t, T]\) with \(\lim_{s \downarrow t} u(s) = v\),

\[
\rho \Gamma(\tau, \phi) - \frac{\partial \Gamma}{\partial \tau}(\tau, \phi) - A^v \Gamma(\tau, \phi) - L(\tau, \phi, v) \geq \delta
\] (26)

for all \((\tau, \phi) \in N(t, \psi)\), where \(N(t, \psi)\) is a neighborhood of \((t, \psi)\). Let \(u(\cdot) \in U[t, T]\) with \(\lim_{s \downarrow t} u(s) = v\), and \(t_1\) such that for \(t \leq s \leq t_1\), the solution \(X(s; t, \psi, u(\cdot)) \in N(t, \psi)\). Therefore, for any \(s \in [t, t_1]\), we have almost surely

\[
\rho \Gamma(s, X_s(t, \psi, u(\cdot))) - \frac{\partial \Gamma}{\partial t}(s, X_s(t, \psi, u(\cdot))) - A^v \Gamma(s, X_s(t, \psi, u(\cdot))) - L(s, X_s(t, \psi, u(\cdot)), u(s)) \geq \delta.
\] (27)

On the other hand, since \(\Gamma \geq V\), using the definition of \(J\) and \(V\), we can get

\[
J(t, \psi; u(\cdot)) \leq \mathbb{E} \left[ \int_t^{t_1} e^{-\rho(s-t)} L(s, X_s, u(s)) ds + e^{-\rho(t_1-t)} V(t_1, X_{t_1}) \right],
\]

13
\[ \leq \mathbb{E} \left[ \int_t^{t_1} e^{-\rho(s-t)} L(s, X_s, u(s)) ds + e^{-\rho(t_1-t)} \Gamma(t_1, X_{t_1}(t, \psi, u(\cdot))) \right]. \]

Using (27) we have
\[
J(t, \psi; u(\cdot)) \leq \mathbb{E} \left[ \int_t^{t_1} e^{-\rho(s-t)} \left( -\delta + \rho \Gamma(s, X_s(t, \psi, u(\cdot))) - \frac{\partial \Gamma}{\partial t}(s, X_s(t, \psi, u(\cdot))) - A u(s) \Gamma(s, X_s(t, \psi, u(\cdot))) \right) ds + e^{-\rho(t_1-t)} \Gamma(t_1, X_{t_1}(t, \psi, u(\cdot))) \right].
\]

In addition, similar to (20), we can get
\[
\mathbb{E} \left[ e^{-\rho(t_1-t)} \Gamma(t_1, X_{t_1}(t, \psi, u(\cdot))) \right] - \Gamma(t, \psi)
= \mathbb{E} \left[ \int_t^{t_1} e^{-\rho(s-t)} \left( \frac{\partial \Gamma(s, X_s(t, \psi, u(\cdot)))}{\partial s} + A u(s) \Gamma(s, X_s(t, \psi, u(\cdot))) \right) ds \right].
\]

Therefore, by virtue of (28), (29), we can get
\[
J(t, \psi; u(\cdot)) \leq - \int_t^{t_1} e^{-\rho(s-t)} \delta ds + \Gamma(t, \psi)
= - \frac{\delta}{\rho} (1 - e^{-\rho(t_1-t)}) + V(t, \psi)
\]

Taking the supremum over all admissible control \( u(\cdot) \in \mathcal{U}[t, T] \), we have
\[
V(t, \psi) \leq - \frac{\delta}{\rho} (1 - e^{-\rho(t_1-t)}) + V(t, \psi).
\]

This contradicts the fact that \( \delta > 0 \). Therefore \( V(t, \psi) \) is a viscosity subsolution. This completes the proof of the theorem. \( \square \)

## 5 Uniqueness

In this section, we will show the uniqueness result for the viscosity solution of the HJB equation (13). Given the uniqueness result, combining the results
we obtained in last section, we can say that the value function \( V(t, \psi) \) is the only viscosity solution of the HJB equation (13).

Since a viscosity solution is both a subsolution and a supersolution, the uniqueness result will follow immediately after establishing the following comparison principle:

**Theorem 5.1 (Comparison Principle)** Assume that \( V_1(t, \psi) \) and \( V_2(t, \psi) \) are both continuous with respect to the argument \((t, \psi)\) and are respectively viscosity subsolution and supersolution of (13) with at most a polynomial growth. In other terms, there exists a real number \( \Lambda > 0 \) and a positive integer \( k > 0 \) such that,

\[
|V_i(t, \psi)| \leq \Lambda(1 + \|\psi\|_2)^k, \quad \text{for} \quad (t, \psi) \in [0, T] \times C, \ i = 1, 2.
\]

Then

\[
V_1(t, \psi) \leq V_2(t, \psi) \quad \text{for all} \quad (t, \psi) \in [0, T] \times C.
\] (31)

Before we go to the proof of the theorem, first let us establish some results which will be needed in the proof. Now let \( V_1 \) and \( V_2 \) be respectively viscosity subsolution and supersolution of (13). For any \( 0 < \delta < 1 \) and \( 0 < \gamma < 1 \), we define

\[
\Theta_{\delta\gamma}(t, \psi, \phi) \equiv \frac{1}{a(\delta)} \|\psi - \phi\|_2^2 \\
+ b(\gamma)e^{(T-t)} \left[ \exp(1 + \|\psi\|_2^2) + \exp(1 + \|\phi\|_2^2) \right],
\] (32)

and

\[
\Phi_{\delta\gamma}(t, \psi, \phi) \equiv V_1(t, \psi) - V_2(t, \phi) - \Theta_{\delta\gamma}(t, \psi, \phi) + M,
\] (33)

where \( M \) is a constant and \( a(x) > 0 \) and \( b(x) > 0 \) are bounded continuous functions such that

\[
\lim_{x \to 0} a(x) = 0 = \lim_{x \to 0} b(x).
\] (34)

Moreover, using the polynomial growth condition for \( V_1 \) and \( V_2 \), the constant \( M \) and the functions \( a \) and \( b \) are chosen such that the maximum value of \( \Phi_{\delta\gamma} \) is zero and

\[
\Phi_{\delta\gamma}(t, \psi, \phi) \leq 0, \quad \text{for each} \quad t \in [0, T], \psi, \phi \in C,
\]

and

\[
\lim_{\|\psi\|_2 + \|\phi\|_2 \to \infty} \Phi_{\delta\gamma}(t, \psi, \phi) = -\infty, \quad \text{for each} \quad t \in [0, T].
\] (35)
Let us denote by \((t_{\delta\gamma}, \psi_{\delta\gamma}, \phi_{\delta\gamma})\) the global maximum of \(\Phi_{\delta\gamma}\). Therefore, we have
\[
\Phi_{\delta\gamma}(t_{\delta\gamma}, \psi_{\delta\gamma}, \phi_{\delta\gamma}) = 0.
\]
Now let us establish a boundedness result for \(\psi_{\delta\gamma}, \phi_{\delta\gamma}\).

**Lemma 5.2** Given \(0 < \delta < 1\) and \(0 < \gamma < 1\), let \((t_{\delta\gamma}, \psi_{\delta\gamma}, \phi_{\delta\gamma})\) be the global maximum of \(\Phi_{\delta\gamma}\). For any \(\gamma > 0\) we can find a constant \(\Lambda_{\gamma} > 0\) such that
\[
\|\psi_{\delta\gamma}\|_2 + \|\phi_{\delta\gamma}\|_2 \leq \Lambda_{\gamma}.
\]  

**Proof.** Observe that
\[
\Phi_{\delta\gamma}(t_{\delta\gamma}, \psi_{\delta\gamma}, \psi_{\delta\gamma}) + \Phi_{\delta\gamma}(t_{\delta\gamma}, \phi_{\delta\gamma}, \phi_{\delta\gamma}) \leq 2\Phi_{\delta\gamma}(t_{\delta\gamma}, \psi_{\delta\gamma}, \phi_{\delta\gamma}).
\]
It implies
\[
V_1(t_{\delta\gamma}, \psi_{\delta\gamma}) - V_2(t_{\delta\gamma}, \psi_{\delta\gamma}) - 2b(\gamma)e^{(T-t_{\delta\gamma})}(\exp(1 + \|\psi_{\delta\gamma}\|_2^2))
+
V_1(t_{\delta\gamma}, \phi_{\delta\gamma}) - V_2(t_{\delta\gamma}, \phi_{\delta\gamma}) - 2b(\gamma)e^{(T-t_{\delta\gamma})}(\exp(1 + \|\phi_{\delta\gamma}\|_2^2))
\leq
2V_1(t_{\delta\gamma}, \psi_{\delta\gamma}) - 2V_2(t_{\delta\gamma}, \phi_{\delta\gamma}) - \frac{2}{a(\delta)}\|\psi_{\delta\gamma} - \phi_{\delta\gamma}\|_2^2
\]
\[\quad - 2b(\gamma)e^{(T-t_{\delta\gamma})}(\exp(1 + \|\psi_{\delta\gamma}\|_2^2) + \exp(1 + \|\phi_{\delta\gamma}\|_2^2)).\]

From the above inequality, it is easy to get that
\[
\frac{2}{a(\delta)}\|\psi_{\delta\gamma} - \phi_{\delta\gamma}\|_2^2 \leq (V_1(t_{\delta\gamma}, \psi_{\delta\gamma}) - V_1(t_{\delta\gamma}, \phi_{\delta\gamma})) + (V_2(t_{\delta\gamma}, \psi_{\delta\gamma}) - V_2(t_{\delta\gamma}, \phi_{\delta\gamma})).
\]  

(37)

By the polynomial growth condition of \(V\), there exists a constant \(\Lambda\) such that
\[
\frac{2}{a(\delta)}\|\psi_{\delta\gamma} - \psi_{\delta\gamma}\|_2^2 \leq \Lambda(1 + \|\psi_{\delta\gamma}\|_2^2 + \|\phi_{\delta\gamma}\|_2^2)^k,
\]
for some positive integer \(k\). So
\[
\|\psi_{\delta\gamma} - \phi_{\delta\gamma}\|_2^2 \leq a(\delta)\Lambda(1 + \|\psi_{\delta\gamma}\|_2 + \|\phi_{\delta\gamma}\|_2)^k.
\]

(38)

Since \((t_{\delta\gamma}, \psi_{\delta\gamma}, \phi_{\delta\gamma})\) is the global maximum of \(\Phi_{\delta\gamma}\), we can get
\[
\Phi_{\delta\gamma}(t, 0, 0) \leq \Phi_{\delta\gamma}(t_{\delta\gamma}, \psi_{\delta\gamma}, \phi_{\delta\gamma}).
\]

(39)
In addition, by the definition of $\Phi_{\delta\gamma}$ and the polynomial growth condition of $V_1, V_2$, we can get

\[ |\Phi_{\delta\gamma}(t, 0, 0)| \leq \Lambda(1 + \|\psi_{\delta\gamma}\|_2 + \|\phi_{\delta\gamma}\|_2)^k. \]

Therefore, by virtue of (39), we have

\[
\begin{align*}
 b(\gamma)e^{(T-t_{\delta\gamma})} & \left[ \exp(1 + \|\psi_{\delta\gamma}\|_2^2) + \exp(1 + \|\phi_{\delta\gamma}\|_2^2) \right] \\
\leq & \quad V_1(t_{\delta\gamma}, \psi_{\delta\gamma}) - V_2(t_{\delta\gamma}, \phi_{\delta\gamma}) - \frac{1}{a(\delta)}\|\psi_{\delta\gamma} - \phi_{\delta\gamma}\|_2^2 - \Phi(t, 0, 0) \\
\leq & \quad 3\Lambda(1 + \|\psi_{\delta\gamma}\|_2 + \|\phi_{\delta\gamma}\|_2)^k. \quad (40)
\end{align*}
\]

It follows that

\[
\begin{align*}
 b(\gamma)e^{(T-t_{\delta\gamma})} & \left[ \exp(1 + \|\psi_{\delta\gamma}\|_2^2) + \exp(1 + \|\phi_{\delta\gamma}\|_2^2) \right] \\
& \quad \leq \left(1 + \|\psi_{\delta\gamma}\|_2 + \|\phi_{\delta\gamma}\|_2\right)^k \leq 3\Lambda.
\end{align*}
\]

Consequently, there exists $\Lambda_{\gamma}$ such that

\[
\|\psi_{\delta\gamma}\|_2 + \|\phi_{\delta\gamma}\|_2 \leq \Lambda_{\gamma}, \text{ and } t_{\delta\gamma} \in [0, T]. \quad (41)
\]

\[ \square \]

The result of Lemma 5.2 implies that for any fixed $\gamma > 0$, the sets $\{\psi_{\delta\gamma}, \delta > 0\}$ and $\{\phi_{\delta\gamma}, \delta > 0\}$ are bounded by $\Lambda_{\gamma}$ independent of $\delta$. We can extract convergent subsequences (as $\delta \to 0$) which we also denote them by $(\psi_{\delta\gamma})$, $(\phi_{\delta\gamma})$, $(t_{\delta\gamma})$ in the rest of the paper.

Moreover, from the inequality (38), we conclude that, for any $\gamma$, there exist two functions that we denote $\psi_{0\gamma}$ and $\phi_{0\gamma}$ such that

\[
\lim_{\delta \to 0} \psi_{\delta\gamma} = \psi_{0\gamma}, \quad \lim_{\delta \to 0} \phi_{\delta\gamma} = \phi_{0\gamma} \quad \text{and} \quad \psi_{0\gamma} = \phi_{0\gamma}. \quad (42)
\]

Using (37) and the previous limit, we obtain the following lemma:

**Lemma 5.3** For a sequence of $\{\delta, \gamma\}$ such that $(t_{\delta\gamma}, \psi_{\delta\gamma}, \phi_{\delta\gamma})$ converges as $\delta, \gamma \to 0$, we have

\[
\lim_{\gamma, \delta \to 0} \frac{2}{a(\delta)}\|\psi_{\delta\gamma} - \phi_{\delta\gamma}\|_2^2 = 0. \quad (43)
\]

\[ \square \]

Now let introduce a functional $F : \mathbb{C} \to \mathbb{R}$ which is defined by

\[
F(\psi) \equiv \|\psi\|_2^2. \quad (44)
\]
It is not hard to show that the map \( F \) is Fréchet differentiable and its derivative is given by \( DF(u)h = 2(u|h) \). This comes from the fact that
\[
\| \psi + h \|^2 - \| \psi \|^2 = 2(\psi|h) + \| h \|^2.
\]
Moreover, we have
\[
2(\psi + h| \cdot ) - 2(\psi| \cdot ) = 2(h| \cdot).
\]
We deduce that \( F \) is twice differentiable and \( D^2F(u)(h,k) = 2(h|k) \).

From the definition of \( \Theta_{\delta \gamma} \) and the definition of \( F \), we can get that
\[
\Theta_{\delta \gamma}(t, \psi, \phi) = \frac{1}{a(\delta)} F(\psi - \phi) + b(\gamma)e^{(T-t)} \left[ e^{1+F(\psi)} + e^{1+F(\phi)} \right].
\]
The following chain rule, quoted from [21] (Theorem 5.2.5 on page 208), is needed to get the Fréchet derivatives of \( \Theta_{\delta \gamma} \):

**Theorem 5.4 (Chain Rule)** Let \( X, Y, Z \) be real Banach spaces. If \( S : X \to Y \) and \( T : Y \to Z \) are Fréchet differentiable at \( x \) and \( y = S(x) \in Y \), respectively, then \( U = T \circ S \) is Fréchet differentiable at \( x \) and
\[
U'(x) = T'(S(x))S'(x).
\]

Given above chain rule, we can say that \( \Theta_{\delta \gamma} \) is Fréchet differentiable. Actually, for \( h, k \in C \), we can get
\[
D_\psi \Theta_{\delta \gamma}(t, \psi, \phi)(h) = \frac{2}{a(\delta)} (\psi - \phi|h) + 2b(\gamma)e^{(T-t)}e^{1+F(\psi)}(\psi|h), \tag{45}
\]
\[
D_\phi \Theta_{\delta \gamma}(t, \psi, \phi)(k) = \frac{2}{a(\delta)} (\phi - \psi|k) + 2b(\gamma)e^{(T-t)}e^{1+F(\phi)}(\phi|k), \tag{46}
\]
and
\[
D^2_\psi \Theta_{\delta \gamma}(t, \psi, \phi)(h,k) = \frac{2}{a(\delta)} (h|k) + 2b(\gamma)e^{(T-t)}e^{1+F(\psi)}[(\psi|h) \cdot (\psi|k) + (h|k)], \tag{47}
\]
\[
D^2_\phi \Theta_{\delta \gamma}(t, \psi, \phi)(h,k) = \frac{2}{a(\delta)} (h|k) + 2b(\gamma)e^{(T-t)}e^{1+F(\phi)}[(\phi|h) \cdot (\phi|k) + (h|k)]. \tag{48}
\]
Observe that we can extend $D_\psi \Theta_{\delta\gamma}(t, \psi, \phi)$ and $D^2_\psi \Theta_{\delta\gamma}(t, \psi, \phi)$, the first and second order Fréchet derivatives of $\Theta_{\delta\gamma}$ with respect to $\psi$, to the space $C \oplus B$ by setting

$$D_\psi \Theta_{\delta\gamma}(t, \psi, \phi)(h + v1_{(0)}) = \frac{2}{\alpha(\delta)}(\psi - \phi|h + v1_{(0)})$$

$$+ 2b(\gamma)e^{(T-t)e^{1+F(\psi)}}(\psi|h + v1_{(0)})$$

and

$$D^2_\psi \Theta_{\delta\gamma}(t, \psi, \phi)(h + v1_{(0)}, k + w1_{(0)}) = \frac{2}{\alpha(\delta)}(h + v1_{(0)}|k + w1_{(0)})$$

$$+ 2b(\gamma)e^{(T-t)e^{1+F(\psi)}}[(\psi|h + v1_{(0)}) \cdot (\psi|k + w1_{(0)})]$$

$$+ (h + v1_{(0)}|k + w1_{(0)})],$$

for $v, w \in \mathbb{R}^n$. Moreover, it is easy to see that these extensions are continuous for that there exists a constant $\Lambda > 0$ such that

$$|((\psi - \phi|h + v1_{(0)})| \leq \|\psi - \phi\|_2 \cdot \|h + v1_{(0)}\|_2$$

$$\leq \Lambda\|\psi - \phi\|_2(\|h\| + |v|);$$

$$|((\psi|h + v1_{(0)})| \leq \|\psi\|_2 \cdot \|h + v1_{(0)}\|_2$$

$$\leq \Lambda\|\psi\|_2(\|h\| + |v|);$$

$$|((\psi|k + w1_{(0)})| \leq \|\psi\|_2 \cdot \|k + w1_{(0)}\|_2$$

$$\leq \Lambda\|\psi\|_2(\|k\| + |w|);$$

and

$$|((k + w1_{(0)}|h + v1_{(0)})| \leq \|k + w1_{(0)}\|_2\|h + v1_{(0)}\|_2$$

$$\leq \Lambda(\|k\| + |v|)(\|h\| + |v|).$$

Similarly, we can extent the first and second order Fréchet derivatives of $\Theta_{\delta\gamma}$ with respect to $\phi$, to the space $C \oplus B$.

In addition, it is easy to verify that for any $\phi \in C$ and $v, w \in \mathbb{R}^n$ we have

$$\langle \phi|v1_{(0)} \rangle = \int_{-\infty}^{0} \langle \phi(s), v1_{(0)}(s) \rangle ds = 0,$$

$$\langle w1_{(0)}|v1_{(0)} \rangle = \int_{-\infty}^{0} \langle w1_{(0)}(s), v1_{(0)}(s) \rangle ds = 0.$$
Proof Theorem 5.1. We define
\[ \Gamma_1(t, \psi) \equiv V_2(t_{\delta\gamma}, \phi_{\delta\gamma}) + \Theta_{\delta\gamma}(t, \psi, \phi_{\delta\gamma}) - M, \] (57)
and
\[ \Gamma_2(t, \phi) \equiv V_1(t_{\delta\gamma}, \psi_{\delta\gamma}) - \Theta_{\delta\gamma}(t, \psi_{\delta\gamma}, \phi) + M. \] (58)

Recall that
\[ \Phi_{\delta\gamma}(t, \psi, \phi) = V_1(t, \psi) - V_2(t, \phi) - \Theta_{\delta\gamma}(t, \psi, \phi) + M \]
reaches its maximum 0 at \((t_{\delta\gamma}, \psi_{\delta\gamma}, \phi_{\delta\gamma})\). By the definition of \(\Gamma_1, \Gamma_2\), it is easy to verify that
\[ \Gamma_1(t, \psi) \geq V_1(t, \psi), \quad \Gamma_2(t, \phi) \leq V_2(t, \phi) \]
and
\[ V_1(t_{\delta\gamma}, \psi_{\delta\gamma}) = \Gamma_1(t_{\delta\gamma}, \psi_{\delta\gamma}), \quad V_2(t_{\delta\gamma}, \phi_{\delta\gamma}) = \Gamma_2(t_{\delta\gamma}, \phi_{\delta\gamma}). \] (59)

Using definition of the viscosity subsolution we have
\[
\frac{\partial \Gamma_1}{\partial t}(t_{\delta\gamma}, \psi_{\delta\gamma}) - \left( \phi \cdot \nabla \Phi_{\delta\gamma}(t, \psi, \phi) \right)(e_j) 1_{\{0\}} + \left( g(t_{\delta\gamma}, \psi_{\delta\gamma}, u)(e_j) 1_{\{0\}} \right) + \left( \Theta_{\delta\gamma}(t, \psi_{\delta\gamma}, \phi) 1_{\{0\}} \right) \leq 0.
\] (60)

Combined with (50), (49), (55) and (56), the above inequality implies,
\[
\frac{\partial \Gamma_1}{\partial t}(t_{\delta\gamma}, \psi_{\delta\gamma}) - \left( \phi \cdot \nabla \Phi_{\delta\gamma}(t, \psi, \phi) \right)(e_j) 1_{\{0\}} \leq 0.
\] (61)

Similarly, using the definition of the viscosity supersolution, we have
\[
\frac{\partial \Gamma_2}{\partial t}(t_{\delta\gamma}, \phi_{\delta\gamma}) - \left( \psi \cdot \nabla \Phi_{\delta\gamma}(t, \psi, \phi) \right)(e_j) 1_{\{0\}} \leq 0.
\] (62)
By virtue of the same techniques we just used, similar to (61), we can get
\[ \rho V_2(t_{\delta,\gamma}, \phi_{\delta,\gamma}) - S(\Gamma_2)(t_{\delta,\gamma}, \phi_{\delta,\gamma}) \]
\[ - \frac{\partial \Gamma_2}{\partial t}(t_{\delta,\gamma}, \phi_{\delta,\gamma}) - \max_{u \in U}[L(t_{\delta,\gamma}, \phi_{\delta,\gamma}, u)] \geq 0. \]  
(63)

By virtue of (61) and (63), we obtain
\[ \rho(V_1(t_{\delta,\gamma}, \psi_{\delta,\gamma}) - V_2(t_{\delta,\gamma}, \phi_{\delta,\gamma})) \]
\[ \leq S(\Gamma_1)(t_{\delta,\gamma}, \psi_{\delta,\gamma}) - S(\Gamma_2)(t_{\delta,\gamma}, \phi_{\delta,\gamma}) \]
\[ + \frac{\partial \Gamma_1}{\partial t}(t_{\delta,\gamma}, \psi_{\delta,\gamma}) - \frac{\partial \Gamma_2}{\partial t}(t_{\delta,\gamma}, \phi_{\delta,\gamma}) \]
\[ + \max_{u \in U}[L(t_{\delta,\gamma}, \psi_{\delta,\gamma}, u)] - \max_{u \in U}[L(t_{\delta,\gamma}, \phi_{\delta,\gamma}, u)]. \]  
(64)

Moreover, using (57), (58) and the definition of \( \Theta_{\delta,\gamma} \), we can get
\[ \frac{\partial \Gamma_1}{\partial t}(t_{\delta,\gamma}, \psi_{\delta,\gamma}) = -\frac{\partial \Gamma_2}{\partial t}(t_{\delta,\gamma}, \phi_{\delta,\gamma}) \]
\[ = -b(\gamma)e^{(T-t_{\delta,\gamma})}[\exp(1 + \|\psi_{\delta,\gamma}\|^2_2) + \exp(1 + \|\phi_{\delta,\gamma}\|^2_2)]. \]  

Thus from (64), we obtain
\[ \rho(V_1(t_{\delta,\gamma}, \psi_{\delta,\gamma}) - V_2(t_{\delta,\gamma}, \phi_{\delta,\gamma})) \]
\[ \leq S(\Gamma_1)(t_{\delta,\gamma}, \psi_{\delta,\gamma}) - S(\Gamma_2)(t_{\delta,\gamma}, \phi_{\delta,\gamma}) \]
\[ - 2b(\gamma)e^{(T-t_{\delta,\gamma})}[\exp(1 + \|\psi_{\delta,\gamma}\|^2_2) + \exp(1 + \|\phi_{\delta,\gamma}\|^2_2)] \]
\[ + \max_{u \in U}[L(t_{\delta,\gamma}, \psi_{\delta,\gamma}, u)] - \max_{u \in U}[L(t_{\delta,\gamma}, \phi_{\delta,\gamma}, u)]. \]  
(65)

Now recall that (see (59))
\[ V_1(t_{\delta,\gamma}, \psi_{\delta,\gamma}) = \Gamma_1(t_{\delta,\gamma}, \psi_{\delta,\gamma}), \quad V_2(t_{\delta,\gamma}, \phi_{\delta,\gamma}) = \Gamma_2(t_{\delta,\gamma}, \phi_{\delta,\gamma}). \]

Using the linearity of \( S \), we see that
\[ S(\Gamma_1)(t_{\delta,\gamma}, \psi_{\delta,\gamma}) - S(\Gamma_2)(t_{\delta,\gamma}, \phi_{\delta,\gamma}) \]
\[ = S(V_1)(t_{\delta,\gamma}, \psi_{\delta,\gamma}) - S(V_2)(t_{\delta,\gamma}, \phi_{\delta,\gamma}) \]
\[ = S(\Theta_{\delta,\gamma})(t_{\delta,\gamma}, \psi_{\delta,\gamma}, \phi_{\delta,\gamma}) - S(M) \]
\[ = S(\Theta_{\delta,\gamma})(t_{\delta,\gamma}, \psi_{\delta,\gamma}, \phi_{\delta,\gamma}). \]  
(66)

In above derivation, \( S(M) = 0 \) comes from the fact that \( M \) is a constant. Now, using (66) and taking the lim sup on both sides of (65) as \( \delta \) and \( \gamma \) go...
to 0, we obtain
\[
\limsup_{\gamma, \delta \to 0} \rho(V_1(t_{\delta \gamma}, \psi_{\delta \gamma}) - V_2(t_{\delta \gamma}, \phi_{\delta \gamma})) \leq \limsup_{\gamma, \delta \to 0} \left\{ -2b(\gamma)e^{(T-t_{\delta \gamma})} \left[ \exp(1 + \|\psi_{\delta \gamma}\|_2) + \exp(1 + \|\phi_{\delta \gamma}\|_2) \right] \\
+ \max_{u \in U} [L(t_{\delta \gamma}, \psi_{\delta \gamma}, u)] - \max_{u \in U} [L(t_{\delta \gamma}, \phi_{\delta \gamma}, u)] \\
+ S(\Theta_{\delta \gamma})(t_{\delta \gamma}, \psi_{\delta \gamma}, \phi_{\delta \gamma}) \right\}.
\] (67)

We know that as \( \delta, \gamma \downarrow 0 \), we have
\( \Theta_{\delta \gamma}(t_{\delta \gamma}, \psi_{\delta \gamma}, \phi_{\delta \gamma}) \to 0 \).

Since \( S \) is a linear functional, we must have
\[
\lim_{\delta, \gamma \to 0} S(\Theta_{\delta \gamma})(t_{\delta \gamma}, \psi_{\delta \gamma}, \phi_{\delta \gamma}) = 0.
\]

Therefore, by virtue of (42) and above equality, we can get
\[
\limsup_{\gamma, \delta \to 0} \rho(V_1(t_{0 \gamma}, \psi_{0 \gamma}) - V_2(t_{0 \gamma}, \phi_{0 \gamma})) \leq 0. \quad (68)
\]

Since \((t_{\delta \gamma}, \psi_{\delta \gamma}, \phi_{\delta \gamma})\) is maximum of \( \Phi_{\delta \gamma} \), then, for all \( \psi \in C \) and \( t \in [0, T] \), we have
\[
\Phi_{\delta \gamma}(t, \psi, \psi) \leq \Phi_{\delta \gamma}(t_{\delta \gamma}, \psi_{\delta \gamma}, \phi_{\delta \gamma})
\]

Then we can get
\[
V_1(t, \psi) - V_2(t, \psi) - 2b(\gamma)e^{(T-t)} \exp(1 + \|\psi\|^2_2) \\
\leq V_1(t_{\delta \gamma}, \psi_{\delta \gamma}) - V_2(t_{\delta \gamma}, \phi_{\delta \gamma}) - \frac{1}{a(\delta)}\|\psi_{\delta \gamma} - \phi_{\delta \gamma}\|^2_2 \\
- b(\gamma)e^{(T-t_{\delta \gamma})} \left[ \exp(1 + \|\psi_{\delta \gamma}\|_2^2) + \exp(1 + \|\phi_{\delta \gamma}\|_2^2) \right] \\
\leq V_1(t_{\delta \gamma}, \psi_{\delta \gamma}) - V_2(t_{\delta \gamma}, \phi_{\delta \gamma}). \quad (69)
\]

where the last inequality comes from the fact that \( a(\delta) > 0 \) and \( b(\gamma) > 0 \).

By virtue of (34) and (68), when we take the limsup on (69) as \( \delta \) and \( \gamma \) go to zero, we can obtain
\[
V_1(t, \psi) - V_2(t, \psi) \leq \limsup_{\gamma, \delta \to 0} (V_1(t_{0 \gamma}, \psi_{0 \gamma}) - V_2(t_{0 \gamma}, \phi_{0 \gamma})) \leq 0. \quad (70)
\]

This completes the proof. \( \square \)

The uniqueness of the viscosity solution of (13) follows directly from this theorem because any viscosity solution is both viscosity subsolution and supersolution.
Conclusions

This paper investigates an optimal control problem for a general system of stochastic functional differential equations with a bounded delay. An infinite-dimensional HJB equation is derived using a Bellman-type dynamic programming principle. It is shown that the value function is the unique viscosity solution of the HJB equation. Computational issues of the inequality remain unresolved and they are the subjects of our future research.

References


